Ruined Moments in Your Life: How Good Are the Approximations?

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Abstract

In this paper we implement numerical PDE solution techniques to compute the *probability of lifetime ruin* which is the probability that a fixed retirement consumption strategy will lead to financial insolvency under stochastic investment returns and lifetime distribution. This problem is a variant of the classical and illustrious ruin calculation from insurance theory, but adapted to individual circumstances.

Using equity market parameters derived from US-based financial data we conclude that a 65-year-old retiree requires 30 times their desired annual (real) consumption to generate a 95% probability of sustainability, which is equivalent to a 5% probability of lifetime ruin, if the funds are invested in a well-diversified portfolio. The 30-to-1 margin of safety contrasts with the relevant annuity factor for an inflation-linked pension which would generate a *zero* probability of lifetime ruin.

Our paper then goes on to compare the PDE-based values with moment matching and comonotonic-based approximations that have been proposed in the literature. We find the Reciprocal Gamma approximation provides an accurate fit as long as the volatility of the underlying investment return does not exceed $\sigma = 30\%$ per annum, which is consistent with capital market history. At higher levels of volatility the moment matching approximations break down. We also confirm that the comonotonic-based lower bound approximation provides remarkably accurate results, although less so at lower levels of volatility.

Our results should be of interest to academics, practitioners and software developers who are interested in estimating these probabilities, but without resorting to crude simulations.

KEYWORDS: Annuity, Pension, Retirement, Stochastic Present Value

1 Motivation

A number of recent papers in the finance and insurance literature have focused on computing the probability a retiring individual will exhaust their wealth under a fixed consumption strategy while still alive. This quantity has been coined the *lifetime ruin* probability and has been investigated by Khorasanee (1996), Milevsky (1998), Milevsky and Robinson (2000), Albrecht and Maurer (2002), Orszag (2002), Gerrard, Haberman and Vigna (2003), Dus, Mitchell and Maurer (2003) and recently Young (2004) amongst others. A variant of this concept has also been coined the "Hurdle Race Problem" by Vanduffel, Dhaene, Goovaerts and Kass (2003), where the problem is formulated to locate an initial provision for future (retirement) payment obligations. Alternatively, this problem has been analyzed under the label of retirement "PensionMetrics" by Blake, Cairns and Dowd (2003). A variant of this problem has also been explored within the context of Asian options where the literature is quite extensive. See Goovaerts, Dhaene and de Schepper (2000) for a discussion of the problem from the point of view of stochastic present value functions.

Regardless of title, the concept of lifetime ruin is at the core of various commercial software packages that provide retirement advice, and motivated by the continued interest in the topic, our paper goes back to first principles and employs analytic techniques to represent the *probability of lifetime ruin* as the solution to a Partial Differential Equation (PDE). We then use a numerical Crank-Nickolson scheme to solve this second-order linear PDE.

With a rapid algorithm at our disposal, we implement our procedure using equity market parameters derived from US-based inflation-adjusted financial data as reported by Ibbotson Associates (2002). We conclude that a 65-year-old retiree requires 30 times their desired annual (real) consumption to generate a 95% probability of sustainability – which is equivalent to a 5% probability of lifetime ruin – if the funds are invested in a well-diversified equity portfolio earning a real (arithmetic mean) 7% per annum with a standard deviation of 20%.

We provide similar estimates for different ages and under a collection of differing return and volatility assumptions. The 30-to-1 margin of safety can be contrasted with the relevant annuity factor for an inflation-linked income which would generate a *zero* probability of lifetime ruin. Thus, for those retirees who decide to self-annuitize, the lifetime ruin probability can provide a summary *risk metric*.

Our paper then goes on to compare the numerical PDE values with various moment matching and other approximations that have been proposed in the literature to compute the lifetime probability of ruin. We label this our *horse race* with an eye towards testing the robustness of the so-called moment matching methodology – which is explained in the body of the paper – and contrasting with approximations which are based on comonotonicity techniques. Our results indicate that the Reciprocal Gamma (RG) approximation and less so the LogNormal (LN) approximation provide an accurate fit as long as the volatility of the underlying investment return does not exceed $\sigma = 30\%$ per annum, which is consistent with capital market history. At higher levels of volatility the moment matching approximations break down. We also confirm that the comonotonic-based lower bound (CLB) approximation – explained and developed at length in a series of papers by Dhaene, Denuit, Goovaerts, Kaas and Vyncke (2002a, 2002b) – provides remarkably accurate results, although less so at lower levels of volatility.

The remainder of this paper is organized as follows. Our general model is presented in section 2. The elementary PDE theory and techniques are presented in section 3. In section 4, we provide a variety of numerical approximation techniques for the lifetime probability of ruin. We start with the so-called Reciprocal Gamma approximation – which is based on a series of papers by Milevsky (1997, 1998, 1999) and Milevsky and Robinson (2000) – we then illustrate the same technique using the LogNormal approximation and finally we implement the comonotonic lower bound method proposed by Dhaene, Denuit, Goovaerts, Kaas and Vyncke (2002a, 2002b). A broad range of numerical examples are presented in section 5, and the paper concludes in section 6.

2 The Probability of Lifetime Ruin

Without any loss of generality we can scale the problem by assuming a constant consumption rate, taken to be one (real or nominal) for simplicity, with a wealth process that obeys the following stochastic differential equation (SDE):

$$dW_t = (\mu W_t - 1)\, dt + \sigma W_t \, dB_t, \qquad W_0 = w, \tag{1}$$

where μ, σ are the drift and diffusion coefficients and B_t is the Brownian motion driving the process. Note that the net-wealth process defined by equation (1) has a drift $(\mu W_t - 1)$, that *may* become negative if μW_t becomes small enough relative to 1. This, in turn, implies that the process W_t *may* eventually hit zero, in stark contrast to the classical geometric Brownian motion which is bounded away from zero in finite time.

Theorem #1: The net-wealth process W_t, defined by equation (1), can be solved explicitly to yield:

$$W_t = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t} \left[w - \int_0^t e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds \right], \qquad W_0 = w. \tag{2}$$

Proof #1: See the book by Karatzas and Shreve (1992, page 361). The proof requires a basic application of the method of variation of coefficients. The solution can be confirmed by applying Ito's Lemma to equation (2) and thus recovering equation (1). A discrete version of equation (1) and (2) under a flexible consumption pattern has been analyzed by Vanduffel, Dhaene, Goovaerts and Kass (2003).

In this paper we are interested in an efficient numerical procedure that will compute three progressive and distinct *ruin probability* values. The first quantity of interest is defined to be:

$$P_1(w, y, t, T \mid \mu, \sigma) := \Pr[W_T \leq y \mid W_t = w], \tag{3}$$

which is the probability that the net-wealth diffusion process W_T will attain a value less than or equal to y, assuming it starts at a value of $W_t = w$ at time $t \geq 0$.

The second quantity of interest is:

$$P_2(w, y, t, T \mid \mu, \sigma) := \Pr[\inf_{t \leq s \leq T} W_s \leq y \mid W_t = w], \tag{4}$$

which is the probability the process W_t *ever* crosses the level of y during the time $[t, T]$.

Finally, the third quantity of interest – and our main objective – represents the *lifetime ruin probability* which is modelled as follows. Let $\mathbf{T}_x$ denote a future lifetime random variable – independent of W_t – with a distribution that is defined to be Gompertz-Makeham (GM) and is parametrized by three variables,

$$\lambda_{x+t} = \lambda + \frac{1}{b} e^{\left(\frac{x+t-m}{b}\right)}, \tag{5}$$

where x denotes the current age of the individual. By definition of the hazard rate function, we have that:

$$\begin{aligned} 1 - F_x(t) &:= \Pr[\mathbf{T}_x \geq t] = e^{-\int_0^t \lambda_{x+s} ds} \\ &= \exp\left\{-\lambda t + b(\lambda_x - \lambda)(1 - e^{t/b})\right\}, \end{aligned} \tag{6}$$

where $F_x(t)$ is the CDF and $f_x(t)$ is the PDF of the random variable $\mathbf{T}_x$. Roughly speaking, one can think of m as the *mode* of the future lifetime and b as a *scale parameter* of $\mathbf{T}_x$. For example, when $\lambda = 0$ and $m = 80$ and $b = 10$, equation (6) stipulates that the probability a current 65-year-old lives to age 85 is: $\Pr[T_{65} \geq 20] = 0.2404$, but the probability that a current 75-year-old lives to age 85 is: $\Pr[T_{75} \geq 10] = 0.3527$. Naturally, the probability of reaching age 85 increases as the individual grows older.

Note some facts about $\mathbf{T}_x$ which will be used later in the analysis. First,

$$\int_0^\infty (1 - F_x(t))\lambda_{x+t} dt = 1, \tag{7}$$

and therefore a simple application of the chain rule retrieves the convenient relationship:

$$\lambda_{x+t} = \frac{f_x(t)}{1 - F_x(t)}. \tag{8}$$

Another important (and well known) fact of any future lifetime random variable is that:

$$E[\mathbf{T}_x] = \int_0^\infty t f_x(t)dt = \int_0^\infty \Pr[\mathbf{T}_x \geq t]dt = \int_0^\infty (1 - F_x(t))dt \tag{9}$$

Thus, under the above-mentioned parameters of $\lambda = 0$, $m = 80$ and $b = 10$, the life expectancy (median life) at age 65 is 79.18 (79.13) and at age 75 is 83.25 (82.62).

Our third and final probability of ruin is defined as:

$$P_3(w, y, x \mid \lambda, m, b, \mu, \sigma) := \Pr[\inf_{0 \leq s \leq \mathbf{T}_x} W_s \leq y \mid W_0 = w], \tag{10}$$

which is the probability the process will ever 'hit' a value of y while the random variable $\mathbf{T}_x$ is still alive. This is the so-called *probability of lifetime ruin.*

Theorem #2. The net-wealth stochastic process W_t defined by equation (2) obeys the following property:

$$P_2(w, 0, t, T \mid \mu, \sigma) = P_1(w, 0, t, T \mid \mu, \sigma), \qquad \forall\, T \geq 0 \tag{11}$$

In other words, the net-wealth process W_t will not cross $y = 0$ more than once. Once it enters the negative region, it stays there. Note that although we are only focused on a constant consumption profile and by analogy a constant wealth-to-consumption ratio, Theorem #2 could be extended to include non-constant consumption profiles. The justification for this should be evident from the subsequent proof.

Proof #2: Equation (2) contains two parts, an exponential function which is strictly greater than zero, multiplied by a term in square brackets whose sign is indeterminate. Therefore, the process W_t,will be less than or equal to zero (ruin) at some future time T, *if, and only if*, the term in square brackets is less than or equal to zero. In other words,

$$W_T \leq 0 \qquad \Longleftrightarrow \qquad w \leq \int_0^T e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds. \tag{12}$$

On the other hand, the integral term is monotonically non-decreasing with respect to the upper bound of integration T. This means that once it becomes greater than w, it *stays* greater than w. Consequently, we arrive at our result that the probability W_t crosses zero prior to a deterministic time T is equivalent to the probability that $W_T \leq 0$.

Given the result from Theorem #2 applied to any fixed value of T, we can generalize to a relationship involving the lifetime ruin probability P_3. Namely,

$$\Pr[\inf_{0 \leq s \leq \mathbf{T}_x} W_s \leq 0 \mid W_0 = w] = \Pr\left[\int_0^{\mathbf{T}_x} e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds \geq w\right] \tag{13}$$

Our moment matching (MM) methodology will be based on approximating the integral in equation (13) with a *suitably close* random variable that share the first few moments with the true (density unknown) variable. This random variable can be interpreted as the stochastic present value of lifetime consumption of \$1 per annum. More on this later.

The analytic approach which leads to a PDE representation will be based on the following analysis of the problem. Motivated by the structure of $\mathbf{T}_x$ we define a *future ruin time* random variable $\mathbf{R}_w^y$ which captures the amount of time it takes for the net-wealth process W_t to 'die' – which is to hit the value of y – assuming it starts at an initial value of $W_t = w$. Note that $\mathbf{R}_w^y$ is independent of the future lifetime random variable $\mathbf{T}_x$. Using our previous notation the formal definition of $\mathbf{R}_w^y$ satisfies:

$$\Pr[\mathbf{R}_w^y \leq t] := P_2(w, y, 0, t \mid \mu, \sigma). \tag{14}$$

From this perspective it should become clear that the lifetime ruin probability P_3 – which is the focus of our analysis – can be represented as follows:

$$P_3(w, y, x \mid \lambda, m, b, \mu, \sigma) = \Pr[\mathbf{R}_w^y \leq \mathbf{T}_x]. \tag{15}$$

It is the probability that the net-wealth process W_t gets ruined *before* the individual dies. We have now transformed the problem to one of computing the cumulative density function (CDF) of the new random variable $\mathbf{R}_w^y - \mathbf{T}_x$, and evaluating this CDF at zero. And, given the natural independence between $\mathbf{R}_w^y$ and $\mathbf{T}_x$, this becomes a simple exercise in probability *convolutions*.

Akin to the future lifetime random variable, let $G_w(t) = \Pr[\mathbf{R}_w^y \leq t]$ denote the CDF in question. We then define the probability density function (PDF) of $\mathbf{R}_w^y$ via:

$$g_w(t) = \frac{\partial G_w(t)}{\partial t} = \frac{\partial P_2(w, y, 0, t \mid \mu, \sigma)}{\partial t}. \tag{16}$$

Note that for $g_w(t)$ to be a proper density function – so that it integrates to a value of one – we must add a probability mass of $1 - P_2(w, y, 0, \infty \mid \mu, \sigma)$ at $g_w(\infty)$, which is the probability the wealth process W_t *never* hits a value of y. In this way, we obtain:

$$\int_0^\infty g_w(t)dt + (1 - P_2(w, y, 0, \infty \mid \mu, \sigma)) = 0. \tag{17}$$

Finally, note that two independent random variables X_1 and X_2 have respective PDFs of $f_1(x)$ and $f_2(x)$, then the PDF $f_3(x)$ of the sum of these two random variables $X_3 = X_1 + X_2$, is given by

$$f_3(y) = \int_{-\infty}^{\infty} f_1(y - z) f_2(z) dz, \tag{18}$$

which leads to:

$$\Pr[X_3 \leq 0] = \int_{-\infty}^{0} \int_{-\infty}^{\infty} f_1(y - z) f_2(z) dz dy \tag{19}$$

In our case, $f_1(x)$ would denote the PDF of future ruin time random variable $\mathbf{R}_w^y$ and $f_2(x)$ would denote the PDF of the negative value of the future lifetime random variable $-\mathbf{T}_x$. The quantity $\Pr[X_3 \leq 0]$ is precisely the probability of lifetime ruin P_3.

Finally, applying some chain-rule calculus to the right-hand-side of equation (19), we are left with:

$$\Pr[\inf_{0 \leq s \leq \mathbf{T}_x} W_s \leq 0 \mid W_0 = w] = \int_0^\infty g_w(t)(1 - F_x(t))dt. \tag{20}$$

One can heuristically think of the integral as 'adding up' the probability of ruin at t, weighted by the probability the individual will survive to this time. Technically, we should add the value $g_w(\infty)$ to the convolution, but since it is weighted by a zero probability of future lifetime survival, we have omitted this term. Note also that in some literature the symbol $({}_tp_x)$ is used to represent $1 - F_x(t)$, which is the conditional survival probability.

3 P.D.E. Representation and Numerical Methods

The ruin probabilities defined in equation (3) and equation (4) are also known as the transition and exit probabilities. It can be shown that they both satisfy the Kolmogorov backward equation, see for example, Bjork (1998).

$$P_t + (\mu w - 1) P_w + \frac{1}{2}\sigma^2 w^2 P_{ww} = 0 \tag{21}$$

with a terminal condition

$$P(w_T, T) = 1 - H(w_T - y) \tag{22}$$

where $H(w)$ is the Heaviside function and w_T is the wealth at T.

The analytic difference between P_1 and P_2 lies in the relevant boundary condition. For P_2 it is obvious that $P_2 = 1$ at $w \leq y$. On the other hand P_1 is non-zero for all $w > 0$. When $w = 0$, we observe that the process defined by (1) implies $dw_t \leq 0$, thus w will remain negative and the proper boundary condition is $P_1 = 1$ at $w \leq 0$. Finally, when $w \to +\infty$, both P_1 and P_2 approach zero.

The next step is to re-scale the problem and realize that:

$$\Pr[\inf_{t \leq s \leq \mathbf{T}_x} W_s \leq y \mid W_t = w] = \Pr[\inf_{0 \leq s \leq \mathbf{T}_{x+t}} W_s \leq y \mid W_0 = w], \tag{23}$$

and therefore

$$P_3(w, y, x+t \mid \lambda, m, b, \mu, \sigma) = \Pr[\inf_{t \leq s \leq \mathbf{T}_x} W_s \leq y \mid W_t = w] \tag{24}$$

from the original definition of the lifetime ruin probability. On the other hand, we also know that

$$P_3(w, y, x+t \mid \lambda, m, b, \mu, \sigma) = \int_0^\infty P_2(w, y, 0, \tau) f_{x+t}(\tau) d\tau. \tag{25}$$

And, from the definition of $f_x(\tau)$ it can be easily verified that

$$f_{x+t}(\tau) = \frac{f_x(t+\tau)}{1-F_x(t)}. \tag{26}$$

Thus

$$\begin{aligned} P_3(w,y,x+t \mid \lambda,m,b,\mu,\sigma) &= \frac{1}{1-F_x(t)} \int_0^\infty P_2(w,y,0,\tau) f_x(\tau+t) d\tau \\ &= \frac{1}{1-F_x(t)} \int_t^\infty P_2(w,y,t,\tau) f_x(\tau) d\tau. \end{aligned} \tag{27}$$

Some algebraic manipulations leads us to the following expressions for the partial derivatives:

$$\begin{aligned} \frac{\partial P_3}{\partial t} &= -\frac{f_x(t)}{1-F_x(t)} P_3 - P_2(w,y,t,t|\mu,\sigma) f_x(t) + \int_t^\infty \frac{\partial}{\partial t} P_2(w,y,0,\tau|\mu,\sigma) f_x(\tau) d\tau \\ &= \lambda_{x+t} P_3 + \int_t^\infty \frac{\partial}{\partial t} P_2(w,y,0,\tau|\mu,\sigma) f_x(\tau) d\tau, \\ \frac{\partial P_3}{\partial w} &= \int_t^\infty \frac{\partial}{\partial w} P_2(w,y,0,\tau|\mu,\sigma) f_x(\tau) d\tau, \\ \frac{\partial^2 P_3}{\partial w^2} &= \int_t^\infty \frac{\partial^2}{\partial w^2} P_2(w,y,0,\tau|\mu,\sigma) f_x(\tau) d\tau. \end{aligned} \tag{28}$$

Note that we have used the identity $f_x(t) = d(1-F_x(t))/dt = -(1-F_x(t))\lambda_{x+t}$. Thus P_3 satisfies the following backward equation:

$$P\lambda_{x+t} = P_t + (\mu w - 1) P_w + \frac{1}{2}\sigma^2 w^2 P_{ww}, \tag{29}$$

with the following terminal condition:

$$P(w_\infty, \infty) = 1 - H(w_\infty - y), \tag{30}$$

where λ_{x+t} is the hazard function which is defined by equation (5). The PDE in equation (29) has also been derived by Young (2003) within the context of controlling the net-wealth diffusion to minimize the probability of lifetime ruin.

3.1 Ruin Probability when $T \to \infty$

When $T \to \infty$ which can be viewed as the perpetuity case, the solution for (21) is independent of t and given by the following ODE:

$$(\mu w - 1)\frac{\partial P}{\partial w} + \frac{1}{2}\sigma^2 w^2 \frac{\partial^2 P}{\partial w^2} = 0. \tag{31}$$

Note that we have dropped the subscript on P since the equation is the same for both P_1 and P_2. The solution of this ODE can be obtained by integration with respect to w twice, and result in:

$$P = C \int_{1/w}^\infty e^{-av} v^{b-1} dv + D \tag{32}$$

where C and D are two constants, $a = 2/\sigma^2$ and $b = 2\mu/\sigma^2 - 1$.

Applying the boundary conditions for P_1 and P_2 yields

$$P_1 = \Gamma(a/w, b), \quad P_2 = \frac{\Gamma(a/w, b)}{\Gamma(a/y, b)}. \tag{33}$$

where

$$a = \frac{2\mu}{\sigma^2} - 1, \quad b = \frac{2}{\sigma^2}, \tag{34}$$

and

$$\Gamma(a, z) = \int_z^\infty e^{-t} t^{a-1} dt. \tag{35}$$

This closed-form analytic representation for the ruin probability is not new – indeed, it has been 'discovered' by a variety of authors in the actuarial, finance and insurance literature – and simply serves to confirm our PDE representation.

3.2 Numerical Scheme

In equation (21), the ruin probability $P(w, t)$ satisfies a second order linear partial differential equation. We solve this equation by a θ-method which can be written as follows:

$$\begin{aligned} &\frac{P_j^{(n+1)} - P_j^{(n)}}{\delta t} + (\mu w_j - 1)\left(\theta \frac{P_{j^*}^{(n+1)} - P_{j^*-1}^{(n+1)}}{\delta w} + (1-\theta)\frac{P_{j^*}^{(n)} - P_{j*-1}^{(n)}}{\delta w}\right) \\ &+ \frac{\sigma^2 w_j^2}{2}\left(\theta \frac{P_{j+1}^{(n+1)} + P_{j-1}^{(n+1)} - 2P_j^{(n+1)}}{\delta w^2} + (1-\theta)\frac{P_{j+1}^{(n)} + P_{j-1}^{(n)} - 2P_j^{(n)}}{\delta w^2}\right) = 0, \end{aligned} \tag{36}$$

where $P_j^{(n)}$ is a grid function which approximates $P(w, t)$ on the grid points $(w_j,\ t_n)$. A uniform grid with equal spacing δt and δx is used. The parameter θ can be arbitrarily selected, but when $\theta = 1/2$ it corresponds to a second order Crank-Nickolson scheme. An upwind scheme is used for the first order derivative P_w, where the variable j^* is either j or $j + 1$, depending on the sign of the coefficient.

For any implicit method where $0 < \theta \leq 1$, numerical boundary conditions must be provided on the computational boundaries $j = 0$ and $j = J$. This can be derived as:

$$P_0^n = 1, \qquad j = 0 \qquad \text{and} \qquad P_J^n = 0, \qquad j = J. \tag{37}$$

$j = 0$ and $j = J$ correspond to the $w_0 = 0$ and $w_J = W$ which are the boundaries of the truncated computation domain for calculating the probability in equation (3), which is P_1. Likewise, for calculating the probability in equation (4), P_2 and the relevant equation (10) for P_3, we use $j = 0$ and $j = J$ with respect to the $w_0 = y$ and $w_J = W$. There are the boundaries of the truncated computation domain. The terminal condition is:

$$P_j^N = 1 - H(w_j - y). \tag{38}$$

With these boundary conditions and the terminal conditions the discrete equations can be solved by matching from time t_n to t_{n+1}, starting from $n = 0$. At t_{n+1}, the equations for $P_j^{(n+1)}$ can be arranged from equation (36). In this space, we can solve for all the probabilities by iteration. For equation (29), we can apply the same method.

4 Analytic Approximations

4.1 Moment Matching for Deterministic T

Using equation (3), equation (4) and Theorem 2 when the ruin level $y = 0$, we can represent our $P_1 = P_2$ probability as:

$$\begin{aligned} P_1 &= \Pr[W_T \leq 0 \mid W_0 = w] \\ &= P_2 = \Pr[\inf_{0 \leq s \leq T} W_s \leq 0 \mid W_0 = w] \\ &= \Pr\left[w \leq \int_0^T e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds\right], \end{aligned} \tag{39}$$

which is equivalent to the probability that the stochastic present value is greater than w. We therefore define the stochastic present value random variable as:

$$\mathbf{Z}_T = \int_0^T e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds, \tag{40}$$

and attempt to approximate the (unknown) distribution of this random variable $\mathbf{Z}_T$ by an approximating density curve. Once again, the connection between the ruin probability and the stochastic present value function has been recognized by a variety of authors, see for example Milevsky (1997) for references, as well Vanduffel, Dhaene, Goovaerts and Kass (2003) in discrete time under flexible consumption.

The approximating density to the stochastic present value (SPV) will be selected so that it's first two moments are identical to the first two moments of the true random variable $\mathbf{Z}_T$. By constructing the 'approximator' in this way, we hope to create a measure of closeness between the two. While the algebra is somewhat tedious, the first moment of $\mathbf{Z}_T$ is:

$$M_1 = E[\mathbf{Z}_T] = \frac{1}{\mu - \sigma^2} - \frac{e^{-(\mu - \sigma^2)T}}{\mu - \sigma^2}, \tag{41}$$

and second moment is:

$$\begin{aligned} M_2 &= E[\mathbf{Z}_T^2] \\ &= \frac{2}{(\mu - 2\sigma^2)(\mu - \sigma^2)}(1 - e^{-(\mu - \sigma^2)T}) \\ &+ \frac{2}{(\mu - 2\sigma^2)(2\mu - 3\sigma^2)}(e^{-(2\mu - 3\sigma^2)T} - 1) \end{aligned} \tag{42}$$

We refer the interested reader to the appendix in which the detailed integral-based derivation of M_1 and M_2 is presented. Note that as σ increases beyond $\sqrt{\mu/2}$ the second moment M_2 becomes very large due to the exponential integrand. Therefore, the moment-matching approximation starts to deteriorate at those levels of volatility. This problem will also occur when $\mathbf{T}$ is random, but the exact threshold at which the exponent 'blows up' will depend on the parameters of the Gompertz density.

4.2 Moment Matching for Stochastic T

Following the representation derived in equation (13) we now compute the first two moments of the stochastic present value when the terminal horizon is stochastic. In this case the random variable is defined as:

$$\mathbf{Z}_{\mathbf{T}_x} = \int_0^{\mathbf{T}_x} e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds. \tag{43}$$

We intend to 'moment match' the stochastic present value $\mathbf{Z}_{\mathbf{T}_x}$ to both the Reciprocal Gamma (RG) distribution and the LogNormal (LN) distribution. Our assumption remains that the future lifetime random variable is Gompertz-Makeham distributed and is independent of the Brownian motion driving the investment return process. We start with

$$\mathbf{Z}_t = \int_0^t e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds, \tag{44}$$

Using the rules for conditional expectations, we know that:

$$E\left[\mathbf{Z}_{\mathbf{T}_x}\right] = E\left[E\left[\mathbf{Z}_t \mid \mathbf{T}_x = t\right]\right] = E\left[E\left[\mathbf{Z} \mid \mathcal{F}_\infty^B\right]\right], \tag{45}$$

where $\mathcal{F}_\infty^B$ is the sigma field generated by the entire path of the Brownian motion. Using the moment generating function for the normal random variable $E\left[\exp\left\{-\sigma B_s\right\}\right] = \exp\{\frac{1}{2}\sigma^2 s\}$, we obtain that:

$$\begin{aligned}
E\left[\mathbf{Z}_{\mathbf{T}_x}\right] &= E\left[E\left[\int_0^t \left(\exp\left\{(\mu - \frac{1}{2}\sigma^2)s + \sigma B_s\right\}\right)^{-1} ds\right] \mid \mathbf{T}_x = t\right] \\
&= E\left[\int_0^t E\left[\exp\left\{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s\right\}\right] ds \mid \mathbf{T}_x = t\right] \\
&= E\left[\int_0^t \exp\{-(\mu - \frac{1}{2}\sigma^2)s\} E\left[\exp\left\{-\sigma B_s\right\}\right] ds \mid \mathbf{T}_x = t\right] \\
&= E\left[\int_0^t \exp\{-(\mu - \frac{1}{2}\sigma^2)s\} \exp\{\frac{1}{2}\sigma^2 s\} ds \mid \mathbf{T}_x = t\right] \\
&= \int_0^\infty \exp\left\{-\left(\mu - \sigma^2\right) s\right\} {}_s p_x \, ds.
\end{aligned} \tag{46}$$

We define the function,

$$A(\xi \mid m, b, x) := \int_0^\infty \exp\{-\xi s\} \; {}_s p_x ds, \tag{47}$$

which is a form of present value operator. Indeed, after substituting the Gompertz (${}_s p_x$) from equation (6) and changing variables, we get:

$$A(\xi \mid m, b, x) = b \exp\left\{\exp\left\{\frac{x-m}{b}\right\} + (x-m)\xi\right\} \Gamma\left(-b\xi\,, \exp\left\{\frac{x-m}{b}\right\}\right), \tag{48}$$

where $\Gamma(u, v) = \int_v^\infty e^{-t} t^{(u-1)} dt$ once again denotes the incomplete Gamma function. By construction, the term $A(\xi \mid m, b, x)$ in equation (47) coincides with the Gompertz price of a life-annuity under a continuously compounded force of interest ξ. Thus, the expectation of the stochastic present value of lifetime consumption is:

$$M_1 = E\left[\mathbf{Z}_{\mathbf{T}_x}\right] = A(\mu - \sigma^2 \mid m, b, x), \tag{49}$$

which is the first (non-central) moment. The same technique, as detailed in equation (46), can be employed to obtain all higher non-central moments of the stochastic variable $\mathbf{Z}_{\mathbf{T}_x}$. The second moment is:

$$M_2 = E\left[\mathbf{Z}^2_{\mathbf{T}_x}\right] = \left(\frac{2}{\mu - 2\sigma^2}\right)\left(A(\mu - \sigma^2 \mid m, b, x) - A(2\mu - 3\sigma^2 \mid m, b, x)\right). \tag{50}$$

The RG and LN approximation method requires the first two moments.

4.3 Reciprocal Gamma Approximation.

The first and second moments of the Reciprocal Gamma random variable are: $M_1 = 1/(\beta(\alpha - 1))$ and $M_2 = 1/(\beta^2(\alpha-1)(\alpha-2))$ respectively. We can then invert the first two moments and express the 'fitted' variables α and β in terms of M_1 and M_2. They are:

$$\alpha = \frac{2M_2 - M_1^2}{M_2 - M_1^2}, \qquad \beta = \frac{M_2 - M_1^2}{M_2 M_1}, \tag{51}$$

where M_1 and M_2 are taken from equations (49) and (50) when we are approximating P_3 and they are taken from equations (3) and (4) when we are approximating P_2.

In either event, the stochastic variable

$$\mathbf{Z}_\eta = \int_0^\eta \exp\{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s\} ds, \tag{52}$$

can be approximated by the Reciprocal Gamma (RG) density function. When we are examining the fixed horizon we use $\eta = T$ and when we are examining the random lifetime horizon we use $\eta = \mathbf{T}_x$. Thus, the probability of lifetime can be approximated by:

$$\Pr[\inf_{0 \le s \le \mathbf{T}_x} W_s \le 0 \mid W_0 = w] \cong \mathbf{G}(1/w \mid \alpha, \beta) \tag{53}$$

where α and β are defined by equation (51). The justification for the RG approximation derives from the limiting arguments provided by equation (33). We refer the interested reader to Milevsky and Robinson (2000) for a similar and more elaborate discussion of this approximation. The current paper is concerned mainly with robustness issues when compared against the PDE values.

4.4 LogNormal Approximation.

We can approximate the unknown distribution of the random variable $\mathbf{Z}_t$ in equation (40) by the LogNormal (LN) distribution instead of the Reciprocal Gamma distribution. The LogNormal density is ubiquitous in the finance literature and is actually used by many practitioners to approximate stochastic present values. Based on a LogNormal 'approximator' the first two moments of the random variable $\mathbf{Z}_t$ are linked via:

$$M_1 = E[\mathbf{Z}_t] = e^{a+\frac{1}{2}b^2}, \tag{54}$$

and

$$M_2 = E[\mathbf{Z}_t^2] = e^{2a+2b^2}, \tag{55}$$

where a and b are the two free parameters (or degrees of freedom) available for the LN distribution. Our numerical examples which we present and compare in the next section will employ the LN approximation exclusively for the P_2 (fixed T) case and thus by equation (3) and (4) we can obtain yet another approximation:

$$\Pr[W_T \leq 0 \mid W_0 = w] = \Pr[\inf_{0\leq s\leq T} W_s \leq 0 \mid W_0 = w] \cong 1 - \Phi\left[\frac{\ln(w) - a}{b}\right], \tag{56}$$

where Φ is the cumulative distribution function of a standard normal distribution.

4.5 Comonotonic Lower Bound (CLB) Approximation.

A series of papers starting with Goovaerts, Dhaene and de Schepper (2000) and culminating with Dhaene, Denuit, Goovaerts, Kaas and Vyncke (2002a, 2002b) use comonotonicity arguments to obtain upper and lower bounds for the stochastic present value of a series of life contingent payments. We have tested both the comonotonic upper bound (CUB) and lower bound (CLB) and find the CLB to be closest to the PDE values. And, while the above-referenced papers by Dhaene *et al* (2002a, 2002b) should be consulted for greater details, the following is a brief review of their approach.

Let $Y(s) = \delta s + \sigma B(s)$ and $\Lambda = \int_0^t e^{-\delta s} B(s) ds$, then we have that Λ is normally distributed with mean 0 and variance:

$$\sigma_\Lambda^2 = Var[\Lambda] = \frac{1}{2\delta^3} + \frac{3 + 2\delta t - 4e^{\delta t}}{2\delta^3 e^{2\delta t}} \tag{57}$$

where $\delta = \mu - \frac{1}{2}\sigma^2$. If we define $r(s)$ by:

$$r(s) = \frac{cov[Y(s), \Lambda]}{\sigma_\Lambda \sigma \sqrt{s}} = \frac{1}{\sigma_\Lambda \sqrt{s}} \left[\frac{1 - e^{-\delta s}}{\delta^2} - \frac{s e^{-\delta s}}{\delta} \right], \qquad s \leq t. \tag{58}$$

the distribution of the random variable $\mathbf{Z}_T$ in equation (40) can be approximated by a new random variable $\tilde{\mathbf{Z}}_T$ which is defined by:

$$\tilde{\mathbf{Z}}_T := \int_0^T e^{-\delta s - r(s)\sigma\sqrt{s}\Phi^{-1}(\mathcal{U}) + \frac{1}{2}\sigma^2 s(1 - r^2(s))} ds, \tag{59}$$

where Φ is the cumulative distribution function of a standard normal distribution and with $\mathcal{U} = \Phi(\frac{\Lambda - E[\Lambda]}{\sigma_\Lambda})$ standard uniformly distributed.

The survival function of $\tilde{\mathbf{Z}}$ is:

$$\Pr[\tilde{\mathbf{Z}}_T > w] = \Phi(z_w), \tag{60}$$

were z_w is the root of the equation

$$\sum_{i=1}^{n} e^{-\delta i \Delta t - -r(i\Delta t)\sigma\sqrt{i\Delta t} z_w + \frac{1}{2}\sigma^2 i \Delta t(1 - r^2(i\Delta t))} \Delta t = w, \tag{61}$$

and where $\Delta t = T/n$. Using this approach, from equations (3) and (4) we obtain:

$$\Pr[W_T \leq 0 \mid W_0 = w] = \Pr[\inf_{0 \leq s \leq T} W_s \leq 0 \mid W_0 = w] \cong \Pr[\tilde{\mathbf{Z}}_T > w]. \tag{62}$$

Note that this particular approximation technique has only been proposed and implemented within the context of a fixed (non-stochastic) time T, and we therefore only present results for P_2. And, although one can always take mortality-weighted averages at different horizons, it is an open question whether the approximation will be as effective, when the horizon itself is a random variable.

5 Numerical Examples and Comparison

We now have the ability to compute some explicit ruin probabilities as well as comparing the performance of various approximation methods. Note once again that with the moment matching method (and comonotonicity techniques) it is only possible to calculate the ruin probabilities P_2 and P_3 when $y = 0$ since the *stochastic present value* representation is only defined when the ruin is set at zero.

Our first table illustrates the difference between the probability of the net-wealth process hitting the level y at any time prior to maturity (P_2) and the probability of the process being under level y at maturity. The table also illustrates our claim in Theorem 2 that both probabilities are identical when $y = 0$. Of course, when $y \neq 0$, the ruin probability of equation (4) is greater than that of equation (3).

[Table 1 goes here]

Table 1 displays the probability that an individual with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined within 30 years (P_2) or at the end of 30 years (P_1), where ruin is defined as wealth hitting a level of y. The market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$. These parameters are consistent with historical evidence on the behavior of a broad portfolio of common equities during the last 75 years, as reported by Ibbotson Associates (2002)

[Figure 1 goes here]

Figure 1 displays the probability that an individual with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined within 30 years (P_2) or at the end of 30 years (P_1), where ruin is defined as wealth hitting a level of y. Once again, the market parameters for the stochastic process driving wealth is an (arithmetic) mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$. Note that as the value of y gets closer to zero, the two quantities converge in value.

[Figure 2 goes here]

Figure 2 displays the probability that an individual who is 65 years old with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined within his lifetime (P_3), where ruin is defined as wealth hitting a level of y. Note that in this case we are dealing with a stochastic time horizon T as opposed to the deterministic $T = 30$ in the previous two figures. Again, the market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$. The mortality parameters are based on a Gompertz approximation to the *unisex* RP-2000 mortality table compiled by the US-based Society of Actuaries, with $m = 87.8$ and $b = 9.5$. Thus, for example, there is effectively a 100% probability that wealth will be drawn-down by at least 50% and will hit 10 dollars while the individual is still alive.

[Figure 3 goes here]

Figure 3 displays the minimum initial wealth level at various ages, that is needed in order to maintain the lifetime ruin probability at *precisely* 1%, 5% and 10% respectively. Thus, for example, a 70 year old would require $w = 40$ to sustain a 1 dollar per annum consumption rate for life, with a 99% probability, but would only require $w = 27$ to sustain this with a 95% probability. The capital market and mortality parameters are as in Table 2. Intuitively, older ages require less initial wealth to generate the same real (after inflation) income pattern with the same level of *probabilistic confidence*. As a general rule of thumb, age 65 requires 30 times consumption for a 95% confidence level.

5.1 Ruin Probabilities P_1, P_2 and P_3

We now provide some explicit results for the Reciprocal Gamma (RG) approximations and compare those to the (true) PDE values to obtain a measure of discrepancy between the two.

[Figure 4 goes here]

Figure 4 displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the ruin probability (P_2) as a function of investment volatility σ, for differing levels of expected investment return μ. Note that the approximate RG value is always greater than the PDE value; i.e. the approximation overstates the ruin probability – and this gap (bias) is an increasing function of volatility. In this particular case the initial wealth is chosen to be $w = 12$ and the terminal horizon is $T = 20$ years.

[Figure 5 goes here]

Figure 5 displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the ruin probability (P_2) as a function of investment volatility σ, for differing terminal horizons T, and assuming an expected growth rate of $\mu = 12\%$ and an initial wealth of $w = 12$. Once again the approximate RG value is always greater than the PDE value and this gap (bias) is an increasing function of volatility σ. But note that for levels of volatility under $\sigma = 30\%$, the RG approximation produces values that are virtually indistinguishable from the PDE values. This is the basis for our statement that the RG method is a valid approximation to lifetime ruin probabilities for historical levels of equity volatility.

[Figure 6 goes here]

Figure 6 displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the ruin probability (P_2) as a function of investment volatility, for differing levels of initial wealth and assuming a $T = 30$ year horizon and $\mu = 15\%$. Note once again that for levels of volatility under 30%, the RG approximation produces values that are virtually indistinguishable from the PDE values, but at higher levels of volatility that approximation is worse the higher the level of initial wealth.

[Figure 7 goes here]

Figure 7 displays the ruin probability as a function of volatility using the numerical PDE method and the approximate Reciprocal Gamma method assuming an initial wealth of $w = 12$ an expected investment return of $\mu = 12\%$ and a terminal horizon of $T = 30$ years. Note that as the volatility increases beyond 30%, the gap in the estimated versus the precise numerical value increases. At very high levels of volatility, the RG approximation breaks down with the ruin probability being given as an eroneuous100%, when in fact it is much lower. The derivation in the appendix provides greater insight into why σ is so critical to the approximation process.

[Figure 8 goes here]

Figure 8 displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the lifetime ruin probability (P_3) as a function of investment volatility σ, for differing levels of expected investment return μ. Although the discrepancy is an increasing function of volatility, it is a decreasing function of the expected investment return. The calculations assume that initial wealth is $w = 12$ and the individual is 65 years old with (unisex) mortality specified by the Gompertz distribution with $m = 87.8$ and $b = 9.5$. Once again, a higher discounting rate μ creates a better *de facto* fit.

[Figure 9 goes here]

Figure 9 displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the lifetime ruin probability (P_3) as a function of investment volatility, for differing levels of initial wealth. We assume the same parameters as in Figure 8.

[Figure 10 goes here]

Figure 10 displays the lifetime ruin probability for an individual aged 65, as a function of volatility using the numerical PDE method and the approximate Reciprocal Gamma method assuming an initial wealth of $w = 12$ an expected investment return of $\mu = 12\%$.

5.2 A Horse Race: LN, CLB, RG and PDE

In this section we compare and contrast the various approximations that have been described in the earlier sections and examine how they perform when benchmarked against the (true) PDE solution.

[Figure 11 and 12 goes here]

Figure 11 as Figure 12 – which are two sections of a cumulative distribution function – compare the results from a variety of methods for computing the ruin probability (P_2) as a function of initial wealth, assuming a $T = 25$ year time horizon. The capital market assumptions are based on historical estimates of real (after-inflation) returns, which are $\mu = 7\%$ and $\sigma = 20\%$. Notice that the CLB method understates the ruin probability at low levels of initial wealth, but matches almost perfectly at higher levels.

[Figure 13 goes here]

Figure 13 displays the ratio of the various approximations to the precise numerical estimate for the ruin probability (P_2) as a function of initial wealth, assuming a $T = 25$ year time horizon. The capital market assumptions are the same as Figure 11.

[Table 2 goes here]

Table 2 compares the probability than an individual with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined; where ruin is defined as wealth hitting zero – within 25 years (P_2) using the exact PDE method, the Reciprocal Gamma (RG) approximation, the LogNormal (LN) approximation and the Comonotonic Lower Bound (CLB) estimate. The deviation of the three approximation methods from the PDE value is listed in brackets. Note that the market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$, which correspond to long-run historical values for these parameters in real (after-inflation) terms.

[Table 3 goes here]

Table 3 compares the probability than an individual with an initial wealth of $w = 15$ dollars (in contrast to Table 2 that examines the case of $w = 20$) who withdraws 1 dollar per annum, will get ruined; where ruin is defined as wealth hitting zero – within 25 years (P_2) using the exact PDE method, the Reciprocal Gamma (RG) approximation, the LogNormal (LN) approximation and the Comonotonic Lower Bound (CLB) estimate. Notice that the ruin probabilities are uniformly higher the lower the level of initial wealth. The capital market parameters are the same as Table 2.

[Table 4 goes here]

Table 4 compares the probability than an individual with an initial wealth of $w = 10$ dollars (in contrast to Table 2 that examines the case of $w = 20$) who withdraws 1 dollar per annum, will get ruined; where ruin is defined as wealth hitting zero – within 25 years (P_2) using the exact PDE method, the Reciprocal Gamma (RG) approximation, the LogNormal (LN) approximation and the Lower Bound (CLB) estimate. The capital market parameters are the same as Table 2.

[Table 5 goes here]

Table 5 reports a particular result from the CLB approximation assuming different discretization schemes. We start with the case where each period is exactly one year – which is the situation reported by Dhaene et. al. (2002a) – and then show the results for quarterly, monthly, weekly and daily compounding. Note that as one would expect, as n gets large the probabilities converge. We assume a $T = 25$ year time horizon and an initial wealth of $w = 15$. The capital market assumptions are based on historical estimates of real (after-inflation) returns, which are $\mu = 7\%$ and $\sigma = 20\%$.

[Table 6 goes here]

Table 6 compares the RG and CLB approximation when the time horizon $T \to \infty$. This is an interesting case in it's own right, since the limiting distribution is in fact Reciprocal Gamma.

[Figure 14 goes here]

In closing, Figure 14 examines the relationship between the number of simulations n, and the accuracy of the probability results for P_2 when benchmarked against the (true) PDE results. The purpose of Figure 13 is to illustrate the large number of simulations that is needed – and the implicit cost of this time – to obtain results that are relatively close to the PDE values. For example, if $n = 500$ runs and $w = 15$ the probability of ruin P_2 within 25 years is 'off' by 4.6% when using simulation values (which takes 1.5 minutes on a PC of Pentium 4, processor 2.0 G). And, even when we increase the number of simulations to $n = 10000$ (which takes 20 minutes on a PC of Pentium 4, processor 2.0 G) the value of P_2 is still 1.4% away from the true value.

6 Conclusion.

With today's advanced computing power – and the intellectual simplicity of simulation – it is quite easy to fall-back on Monte Carlo techniques to derive all forms of *lifetime ruin* probabilities. This is especially common amongst practitioners who are interested in quick-and-dirty heuristic approximations. In this paper we have shown how to formulate and then numerically solve the PDE representation of the lifetime ruin probability; a quantity which has been investigated by numerous authors in the finance and insurance literature. We find that as a general rule of thumb age 65 requires 30 times consumption for a 95% confidence level, i.e. a 5% probability of lifetime ruin, when historical capital market parameters are assumed.

Using these parameter values we then compared our numerical PDE results with various moment matching and bounding approximations. Our analysis indicates that under realistic growth rate assumptions the Reciprocal Gamma approximation proposed by Milevsky and Robinson (2000) provides an accurate fit as long as the volatility of the underlying investment return does not exceed $\sigma = 30\%$ per annum. These parameters, once again, are consistent with historical capital market values as reported by Ibbotson Associates (2002). However, at higher levels of volatility the RG approximation breaks down. In contrast, the CLB approximation which is based on the work of Dhaene et. al. (2002a, 2002b) is more stable and accurate across all levels of volatility. And, even though CLB tends to underestimate the ruin probability at lower levels of initial wealth, overall we believe it is preferred to any moment matching approximation.

However, the CLB approximation has not yet been implemented for a stochastic lifetime distribution, in contrast to the RG methodology which was specifically developed for this case. Also, the CLB is not applicable when computing the probability of hitting arbitrary levels of wealth – as opposed to zero – since the equality between the probability of ruin and the CDF of the stochastic present value, does not apply.

This leaves fruitful areas for investigation, and as is the tradition, further research will apply the numerical PDE approach to obtain *retirement ruin* probabilities for more complex models of investment returns and consumption strategies.

7 Appendix

In this appendix we provide for reference and confirmation the exact derivation of the first and second moment of the *stochastic present value* random variable defined by $\mathbf{Z}_T$ in equation (40).

$$\mathbf{Z}_T = \int_0^T e^{-(\mu-\frac{1}{2}\sigma^2)s-\sigma B_s} ds,$$

$$\begin{aligned} M_1 = E[\mathbf{Z}_T] &= \int_0^T e^{-(\mu-\frac{1}{2}\sigma^2)t} \int_{-\infty}^{\infty} e^{-\sigma x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dxdt \\ &= \int_0^T e^{-(\mu-\frac{1}{2}\sigma^2)t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x+t\sigma)^2}{2t}+\frac{1}{2}\sigma^2 t} dxdt \\ &= \int_0^T e^{-(\mu-\frac{1}{2}\sigma^2)t} e^{\frac{1}{2}\sigma^2 t} dt \\ &= \begin{cases} \frac{2}{\sigma^2}\left(e^{\frac{1}{2}\sigma^2 T} - 1\right), & \mu - \frac{1}{2}\sigma^2 = 0 \\ T, & \mu - \sigma^2 = 0 \\ \frac{1}{\mu-\sigma^2}\left(1 - e^{-(\mu-\sigma^2)T}\right), & \text{Otherwise} \end{cases} \end{aligned}$$

and, using the same arguments, the second moment is:

$$\begin{aligned} M_2 = E[\mathbf{Z}_T^2] &= E[(\int_0^T e^{-(\mu-\frac{1}{2}\sigma^2)t-\sigma Bt} dt)^2] \\ &= E[\int_0^T e^{-(\mu-\frac{1}{2}\sigma^2)t-\sigma Bt} dt \int_0^T e^{-(\mu-\frac{1}{2}\sigma^2)s-\sigma Bs} ds] \\ &= E[2\int_0^T \int_0^t e^{-(\mu-\frac{1}{2}\sigma^2)t} e^{-(\mu-\frac{1}{2}\sigma^2)s} e^{-\sigma Bt} e^{-\sigma Bs} dsdt \\ &= 2\int_0^T \int_0^t e^{-(\mu-\frac{1}{2}\sigma^2)t} e^{-(\mu-\frac{1}{2}\sigma^2)s} E[e^{-\sigma Bt} e^{-\sigma Bs}] dsdt \end{aligned}$$

We require the covariance term $E[e^{\sigma B_t} e^{\sigma B_s}]$, where $s < t$. In this case, assume $B_t = B_s + W$, where $W \sim N(0, \sqrt{t-s})$. This leads to:

$$\begin{aligned} E[e^{-\sigma B_t} e^{-\sigma B_s}] &= E[e^{-\sigma(B_s+W)} e^{-\sigma B_s}] \\ &= E[e^{-\sigma W} e^{-2\sigma B_s}] \\ &= E[e^{-\sigma W}] E[e^{-2\sigma B_s}] \\ &= e^{\frac{1}{2}\sigma^2(t-s)} e^{2\sigma^2 s} \\ &= e^{\frac{3}{2}\sigma^2 s} e^{\frac{1}{2}\sigma^2 t}. \end{aligned}$$

Substituting into M_2, we can get

$$M_2 = E[\mathbf{Z}_T^2] = 2\int_0^T \int_0^t e^{-(\mu-\frac{1}{2}\sigma^2)t} e^{-(\mu-\frac{1}{2}\sigma^2)s} e^{\frac{3}{2}\sigma^2 s} e^{\frac{1}{2}\sigma^2 t} dsdt$$

$$= \begin{cases} \frac{1}{3\sigma^4}\left(2e^{2\sigma^2 T} - 8e^{\frac{1}{2}\sigma^2 T+6}\right) & \mu - \frac{1}{2}\sigma^2 = 0 \\ \frac{T}{\mu-2\sigma^2} + \frac{e^{-(\mu-2\sigma^2)T}-1}{(\mu-2\sigma^2)^2} & \mu - \sigma^2 = 0 \\ \frac{1-e^{-(\mu-\sigma^2)T}}{(\mu-\sigma^2)^2} - \frac{2Te^{-(\mu-\sigma^2)T}}{\mu-\sigma^2} & \mu - 2\sigma^2 = 0 \\ \frac{2}{(\mu-2\sigma^2)(\mu-\sigma^2)}(1 - e^{-(\mu-\sigma^2)T}) & \\ +\frac{2}{(\mu-2\sigma^2)(2\mu-3\sigma^2)}(e^{-(2\mu-3\sigma^2)T} - 1), & \text{Otherwise} \end{cases}$$

Q.E.D.

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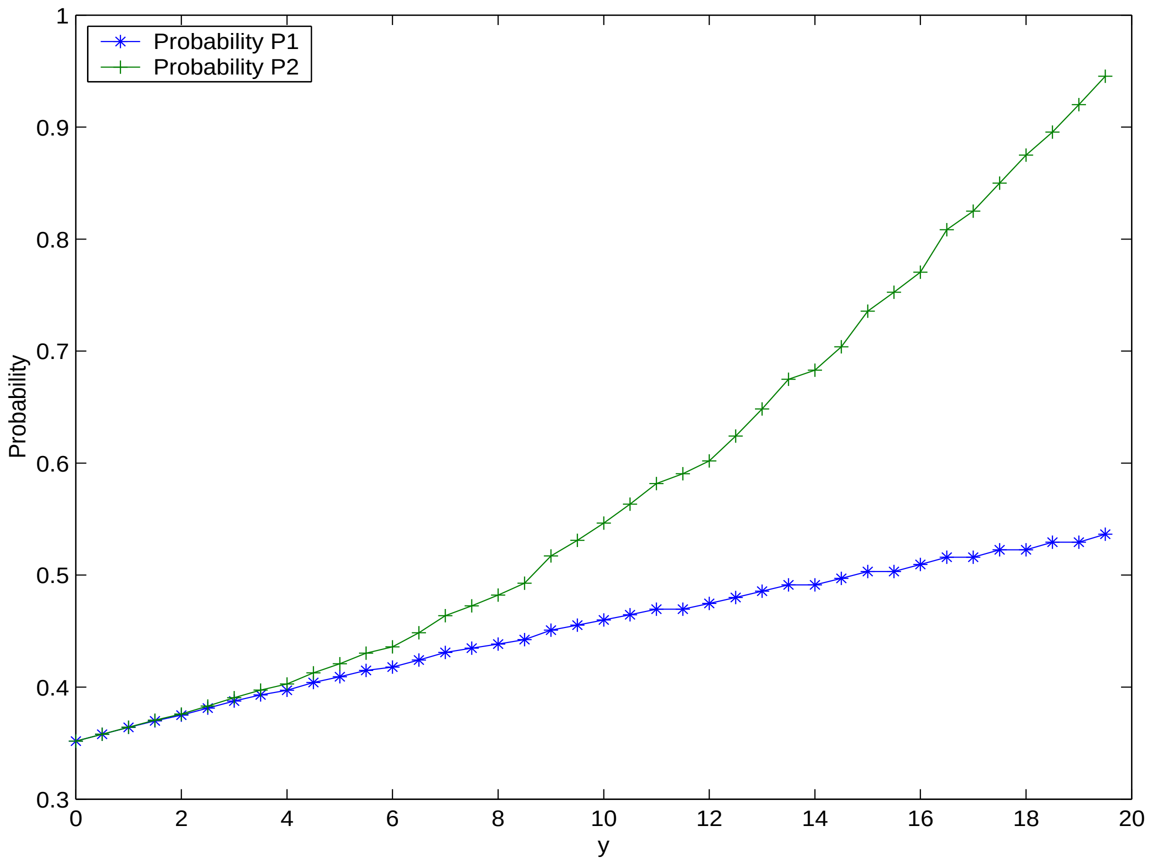

Figure 1: The figure displays the probability that an individual with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined within 30 years (P_2) or at the end of 30 years (P_1), where ruin is defined as wealth hitting a level of y. The market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$

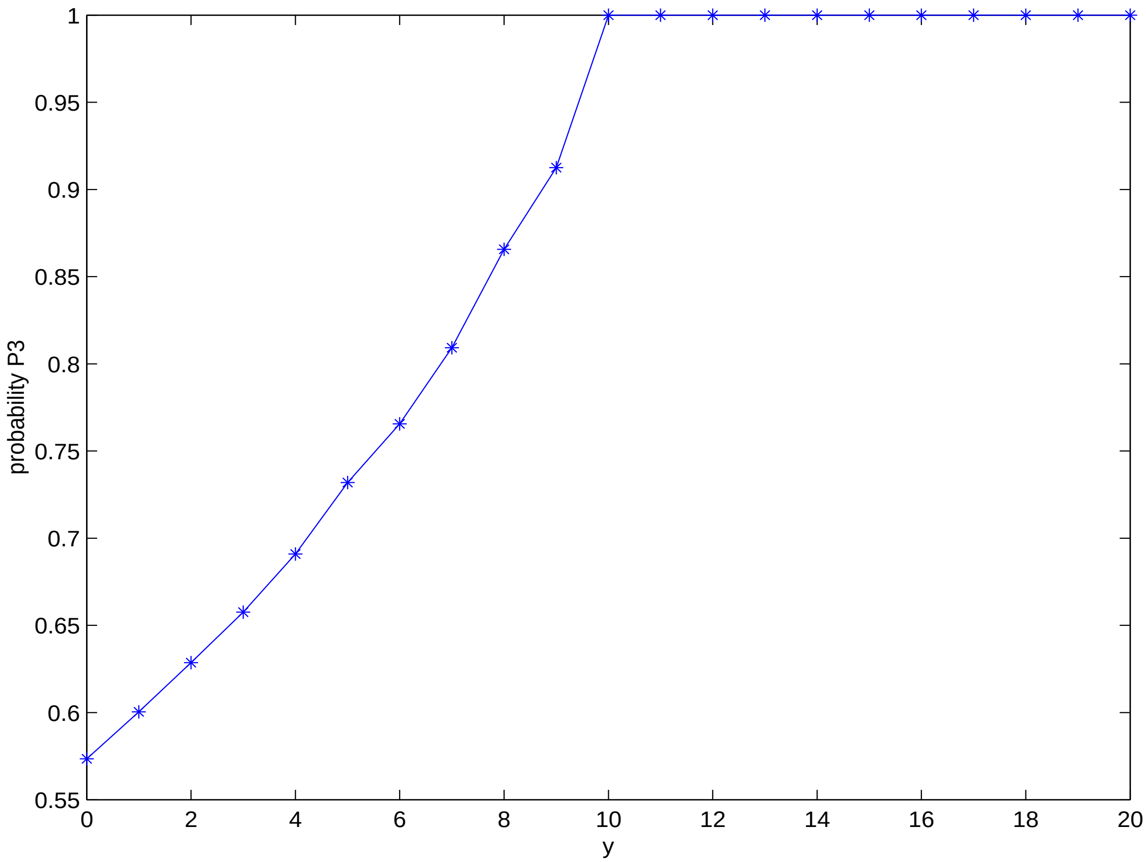

Figure 2: The figure displays the probability that an individual who is 65 years old with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined within his lifetime (P_3), where ruin is defined as wealth hitting a level of y. The market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$. The mortality parameters are based on a Gompertz approximation with $m = 87.8$ and $b = 9.5$. Thus, for example, there is effectively a 100% probability that wealth will be drawn-down; and will hit 10 dollars while the individual is still alive.

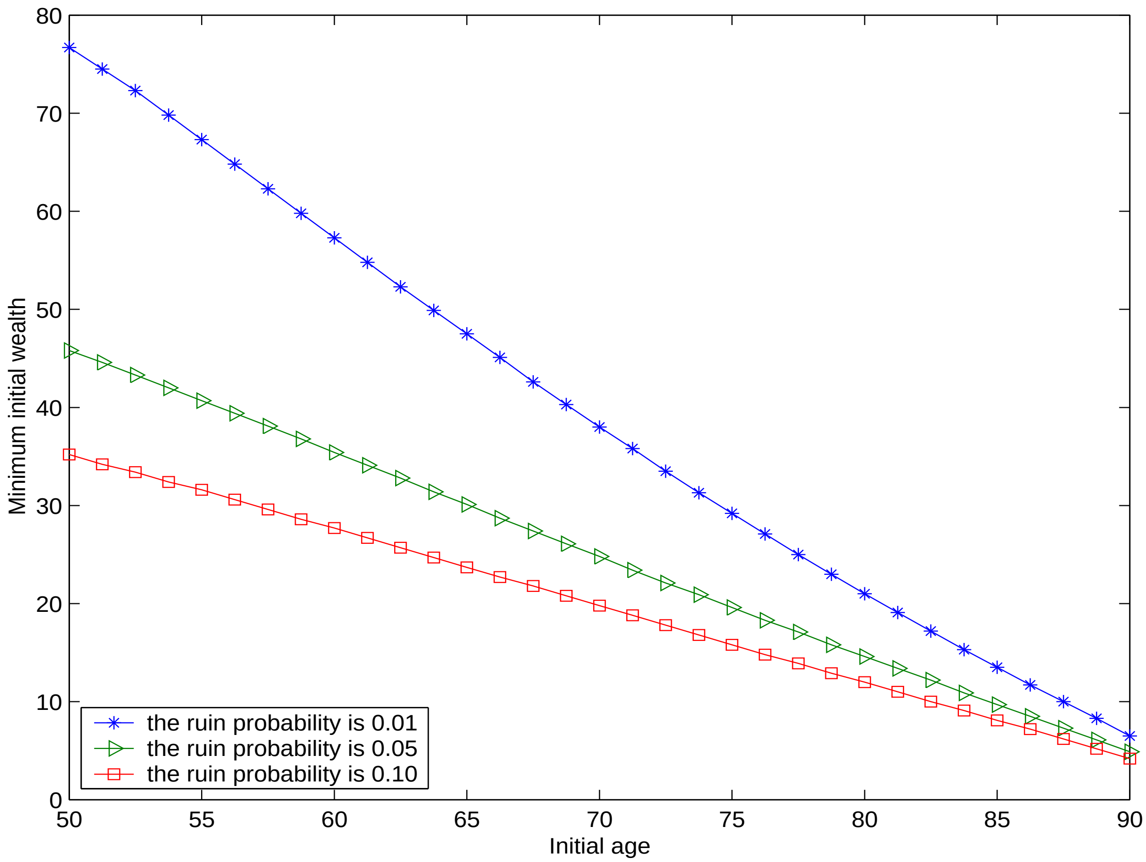

Figure 3: The figure displays the minimum initial wealth level at various ages, that is needed in order to maintain the lifetime ruin probability at 1%, 5% and 10% respectively. Thus, for example, a 70 year old would require $w = 40$ to sustain a 1 dollar per annum consumption rate for life, with a 99% probability, but would only require $w = 27$ to sustain this with a 95% probability. The capital market and mortality parameters are as in Table 2.

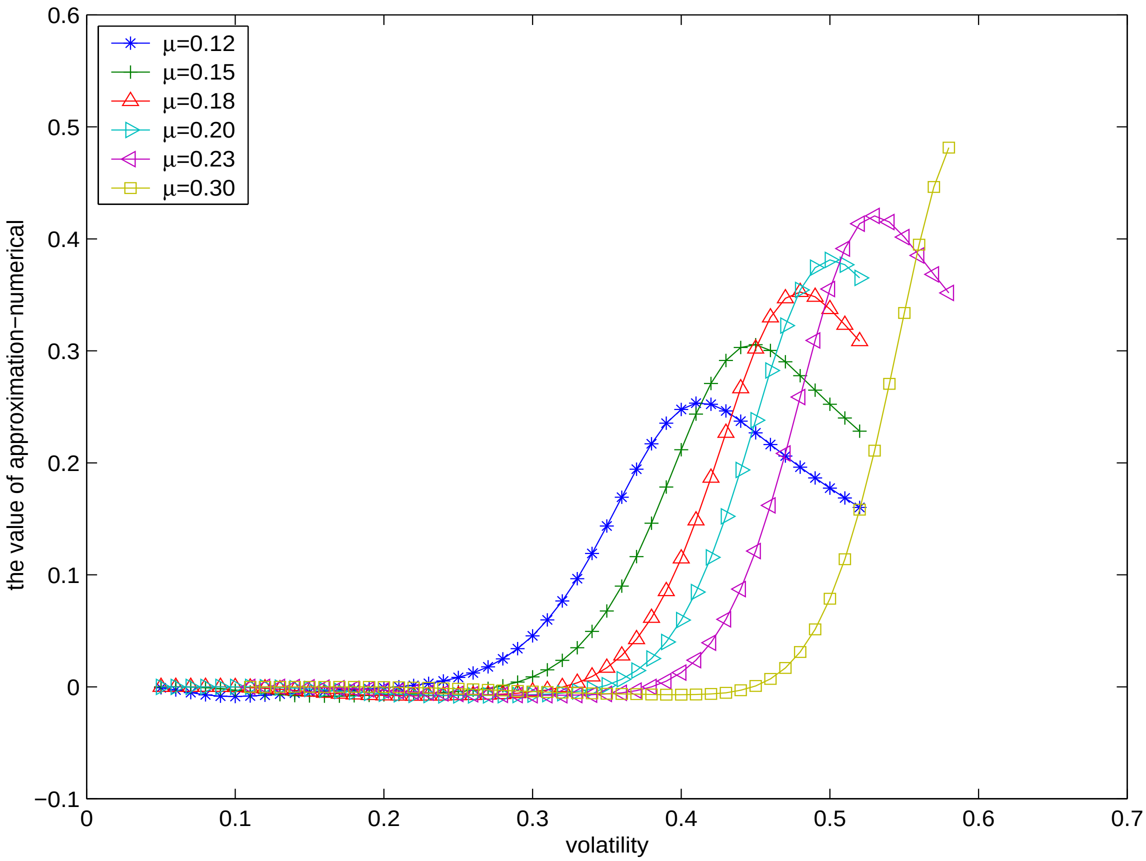

Figure 4: The figure displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the ruin probability (P_2) as a function of investment volatility, for differing levels of expected investment return. Note that the approximate RG value is always greater than the PDE value; i.e. the approximation overstates the ruin probability – and this gap (bias) is an increasing function of volatility. Note the assumption that initial wealth is $w = 12$ and the terminal horizon is $T = 20$ years.

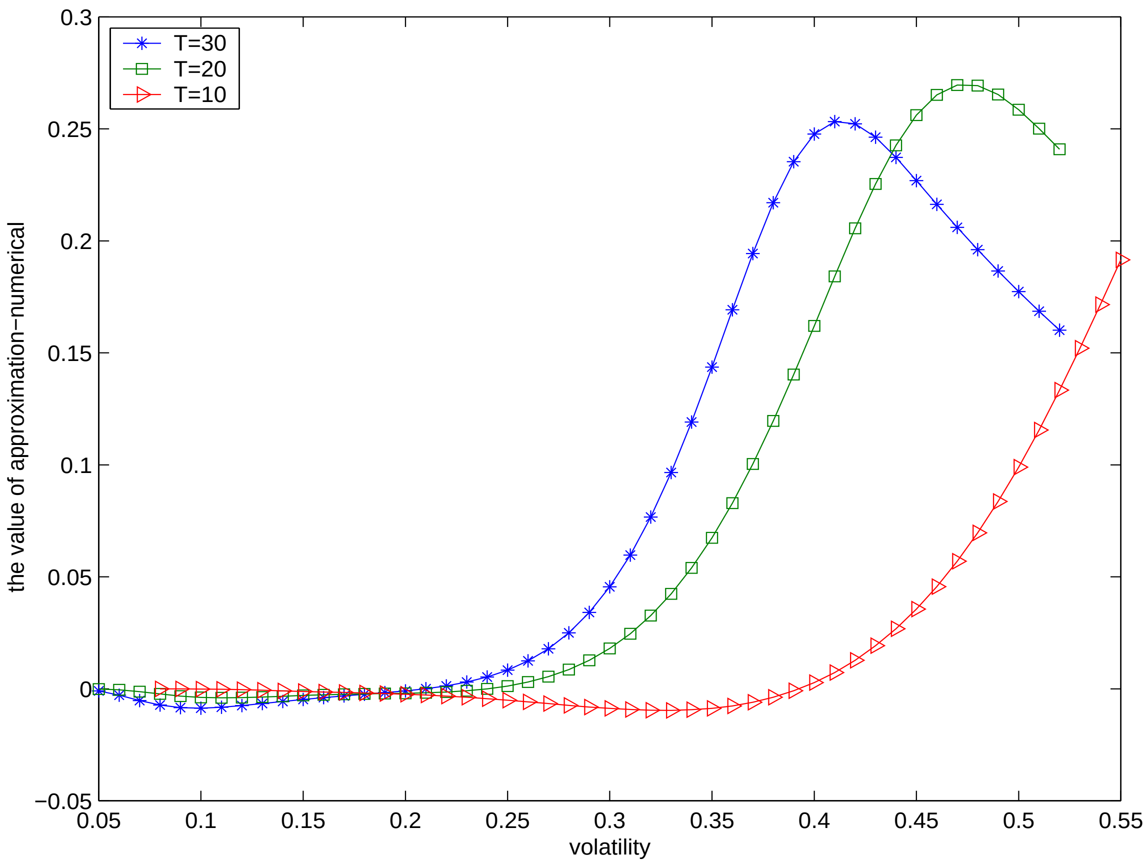

Figure 5: The figure displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the ruin probability (P_2) as a function of investment volatility, for differing terminal horizons and assuming an expected growth rate of $\mu = 12\%$ and an initial wealth of $w = 12$. Once again the approximate RG value is always greater than the PDE value and this gap (bias) is an increasing function of volatility. But note that for levels of volatility under 30%, the RG approximation produces values that are virtually indistinguishable from the PDE values.

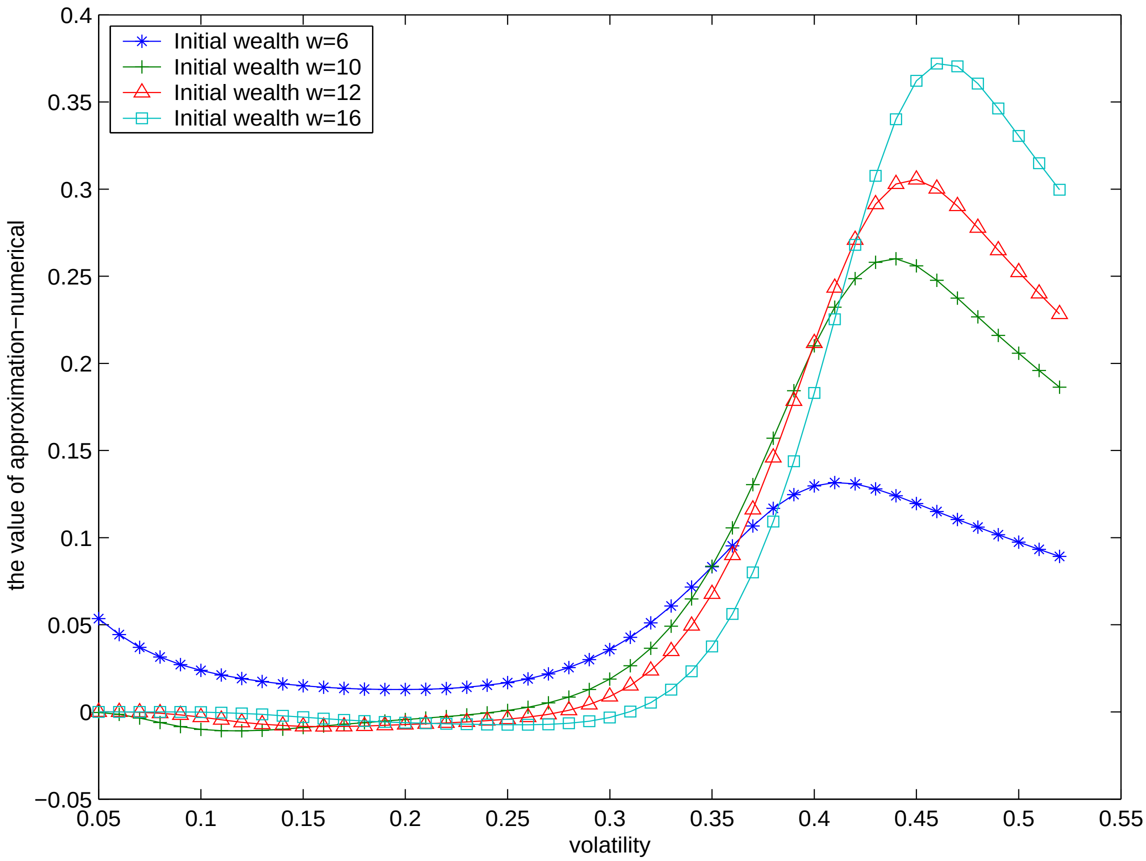

Figure 6: The figure displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the ruin probability (P_2) as a function of investment volatility, for differing levels of initial wealth and assuming a $T = 30$ year horizon and $\mu = 15\%$. Note that for levels of volatility under 30%, the RG approximation produces values that are virtually indistinguishable from the PDE values, but at higher levels of volatility that approximation is worse the higher the level of initial wealth.

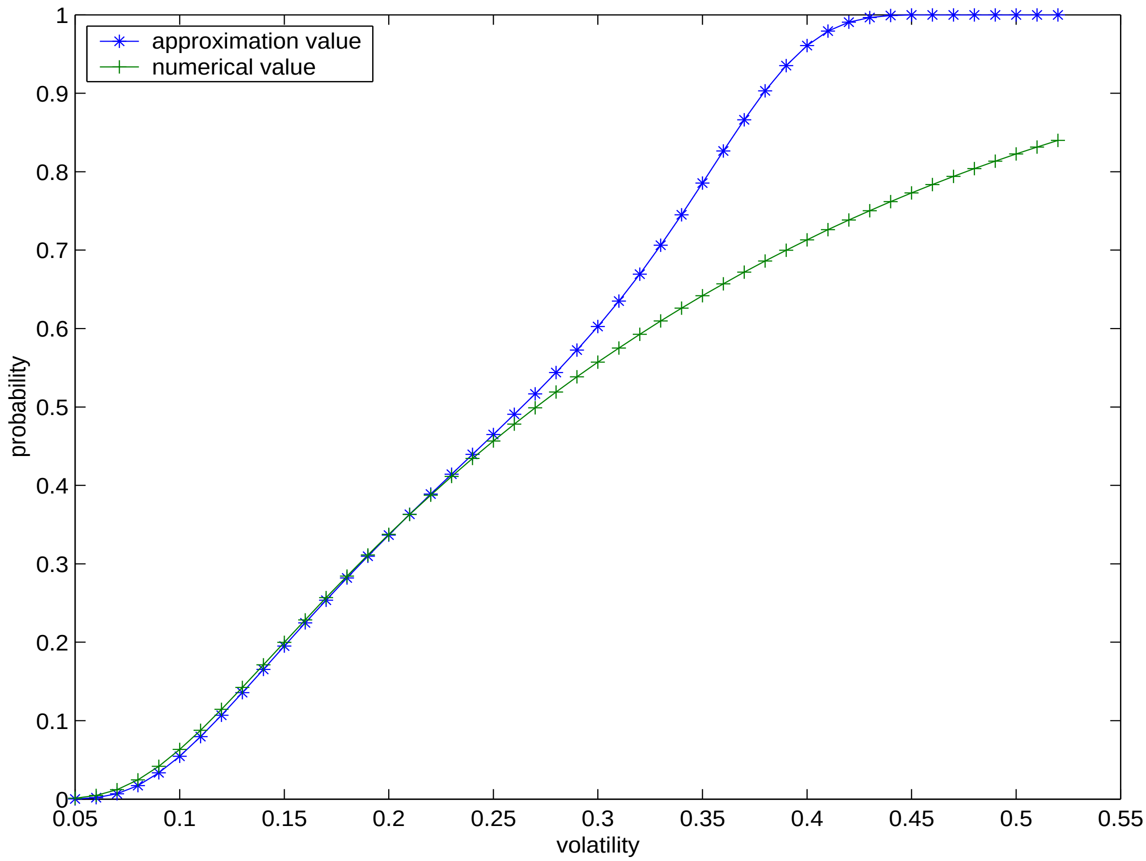

Figure 7: The figure displays the ruin probability as a function of volatility using the numerical PDE method and the approximate Reciprocal Gamma method assuming an initial wealth of $w = 12$ an expected investment return of $\mu = 12\%$ and a terminal horizon of $T = 30$ years. Note that as the volatility increases beyond 30%, the gap in the estimated versus the precise numerical value increases. At very high levels of volatility, the RG approximation breaks down with the ruin probability being given as 100%, which in fact it is much lower.

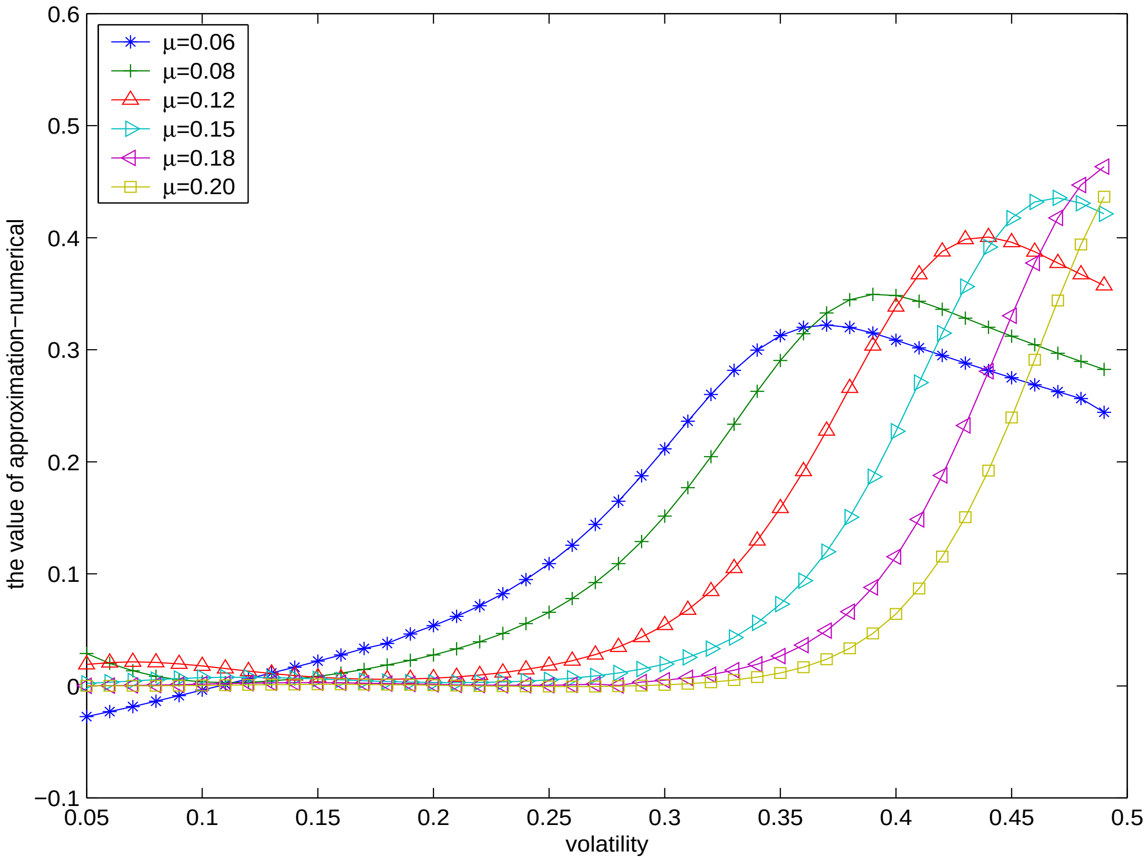

Figure 8: The figure displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the lifetime ruin probability (P_3) as a function of investment volatility, for differing levels of expected investment return. Although the discrepancy is an increasing function of volatility, it is a decreasing function of the expected investment return. The calculations assume that initial wealth is $w = 12$ and the individual is 65 years old with mortality specified by the Gompertz distribution with $m = 87.8$ and $b = 9.5$.

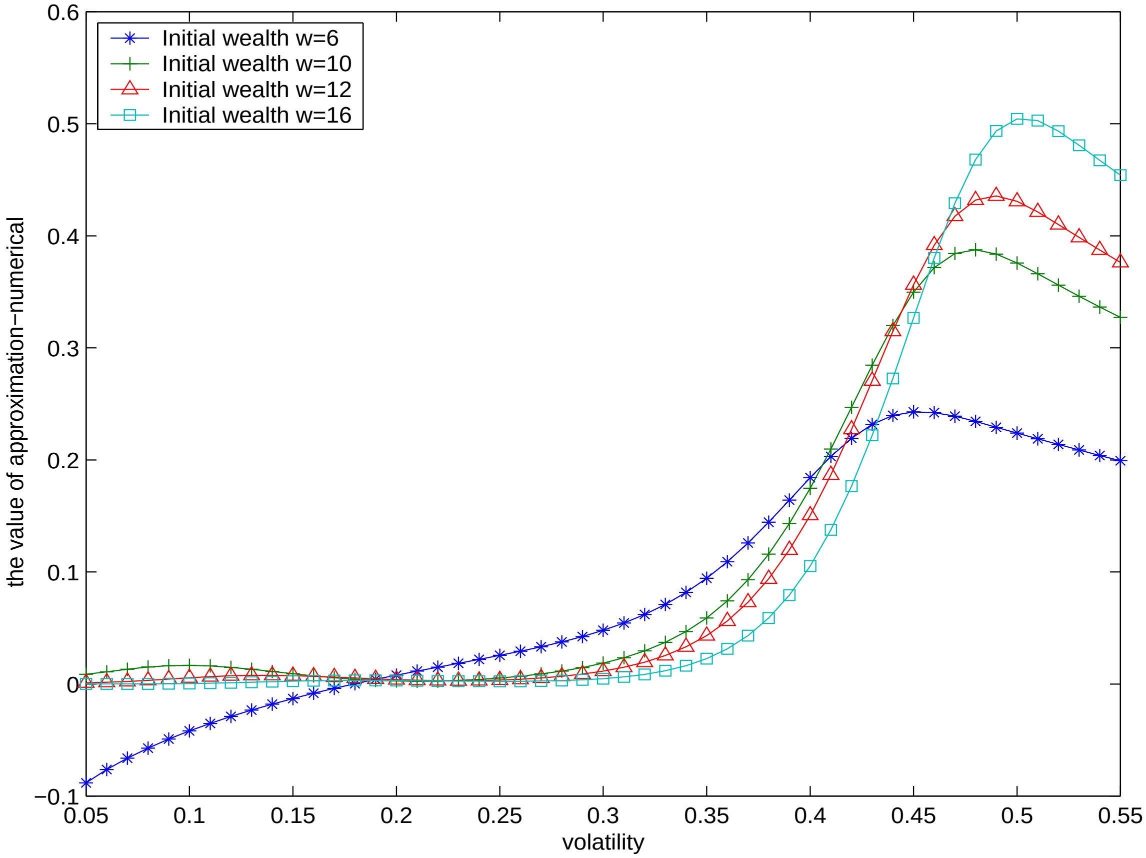

Figure 9: The figure displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the lifetime ruin probability (P_3) as a function of investment volatility, for differing levels of initial wealth. We assume the same parameters as in Figure 8.

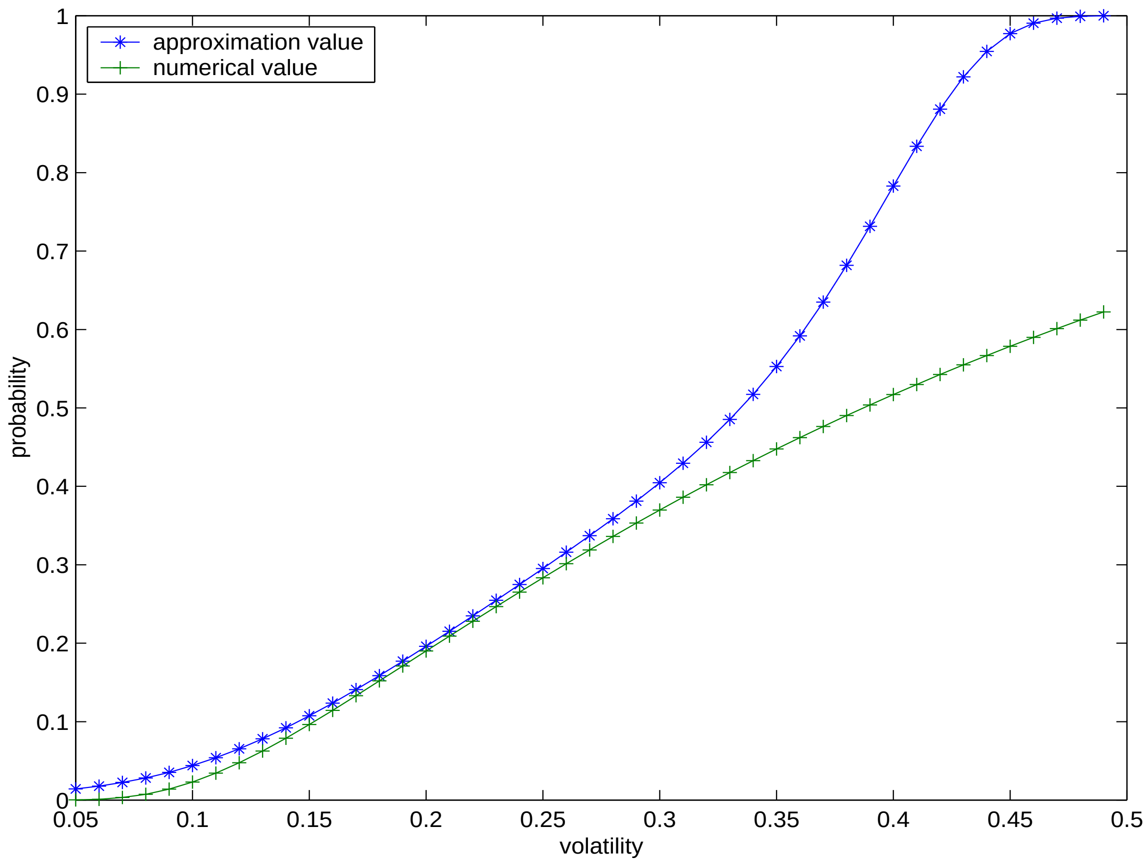

Figure 10: The figure displays the lifetime ruin probability for an individual aged 65, as a function of volatility using the numerical PDE method and the approximate Reciprocal Gamma method assuming an initial wealth of $w = 12$ an expected investment return of $\mu = 12\%$.

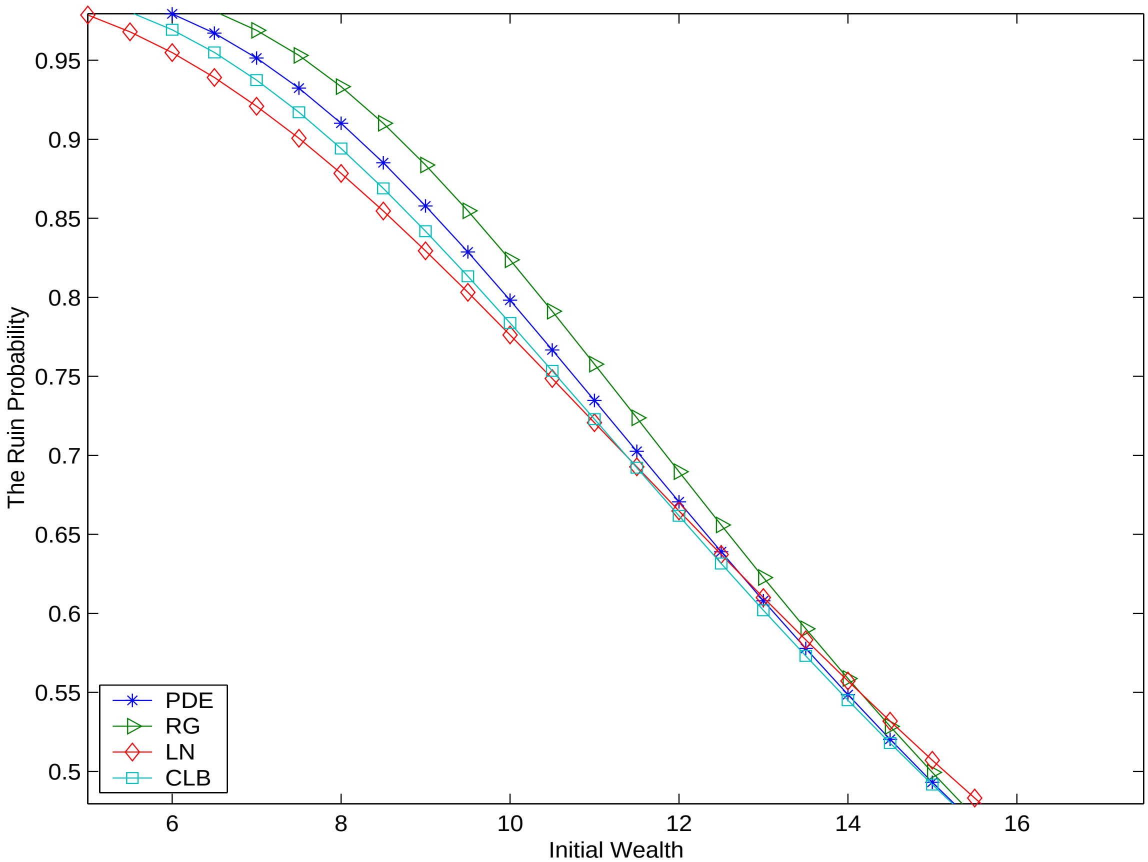

Figure 11: The figure compares the results from a variety of methods for computing the ruin probability (P_2) as a function of initial wealth, assuming a $T = 25$ year time horizon. The capital market assumptions are based on historical estimates of real (after-inflation) returns, which are $\mu = 7\%$ and $\sigma = 20\%$.

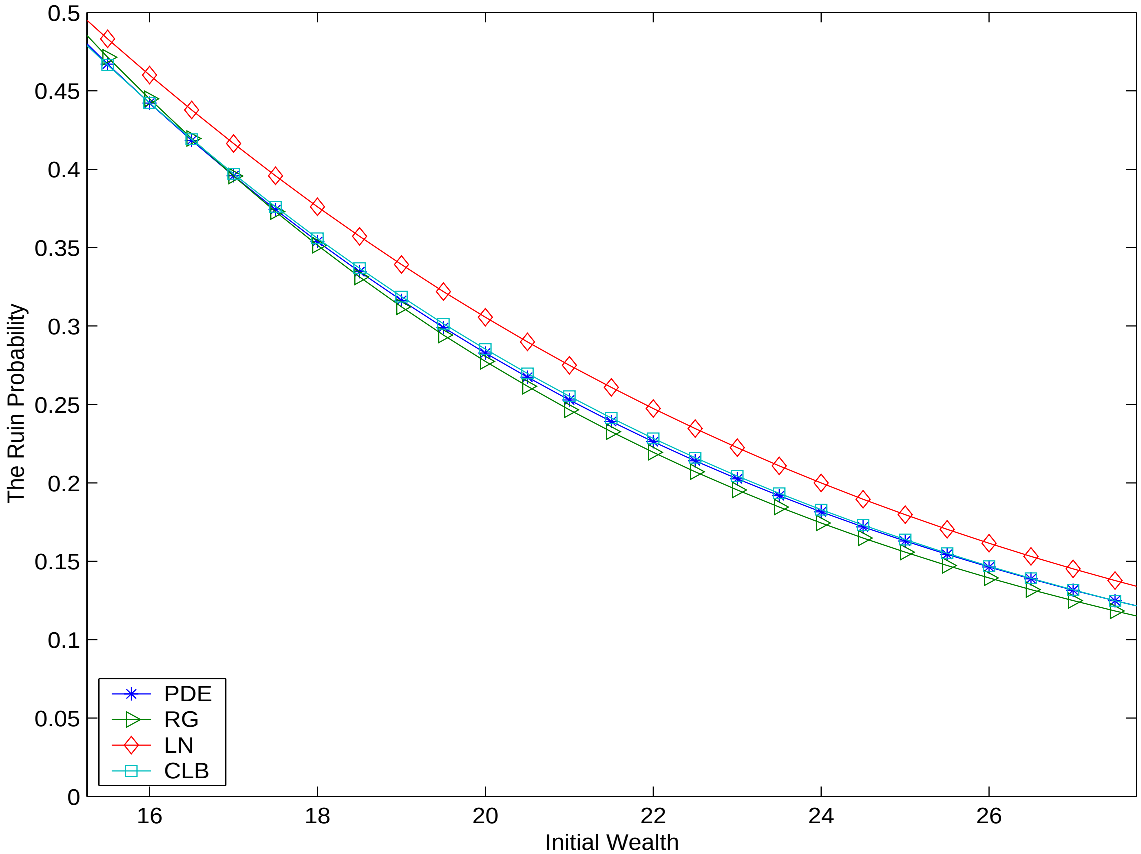

Figure 12: The figure compares the results from a variety of methods for computing the ruin probability (P_2) as a function of initial wealth, assuming a $T = 25$ year time horizon. The capital market assumptions are based on historical estimates of real (after-inflation) returns, which are $\mu = 7\%$ and $\sigma = 20\%$.

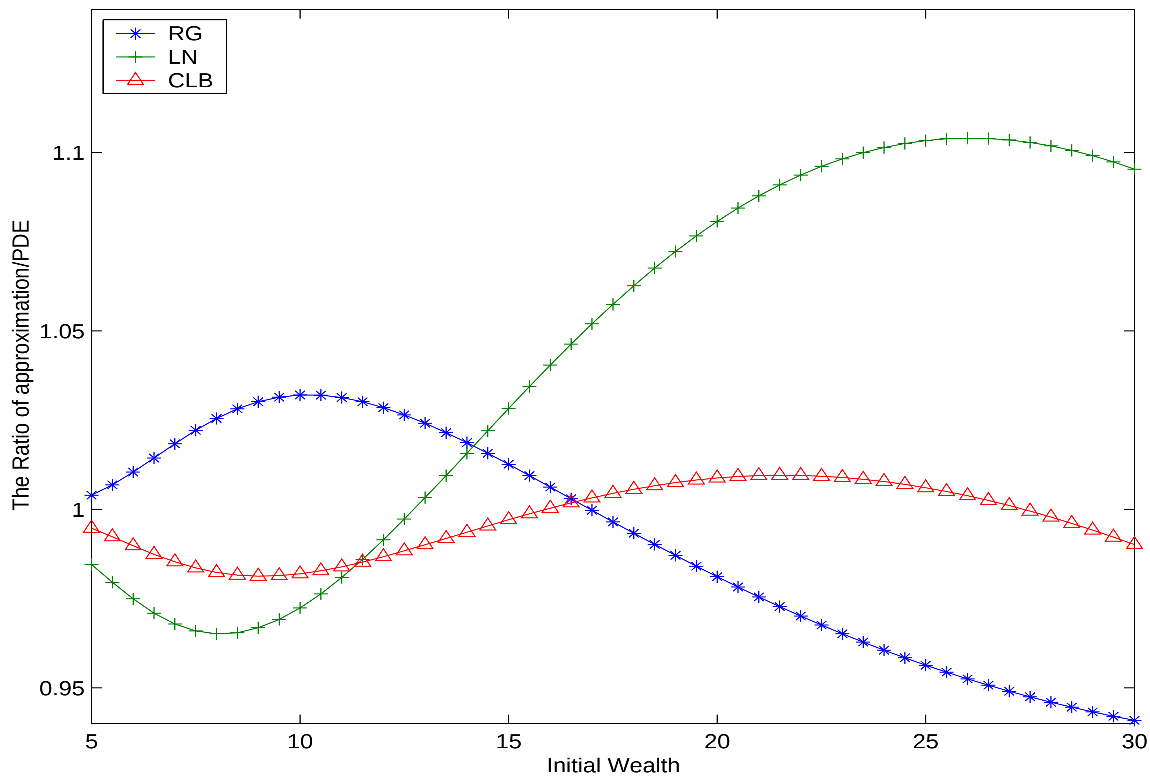

Figure 13: The figure displays the ratio of the various approximations to the precise numerical estimate for the ruin probability (P_2) as a function of initial wealth, assuming a $T = 25$ year time horizon. The capital market assumptions are the same as Figure 11. The best approximation is the curve closest to 1 at all wealth levels.

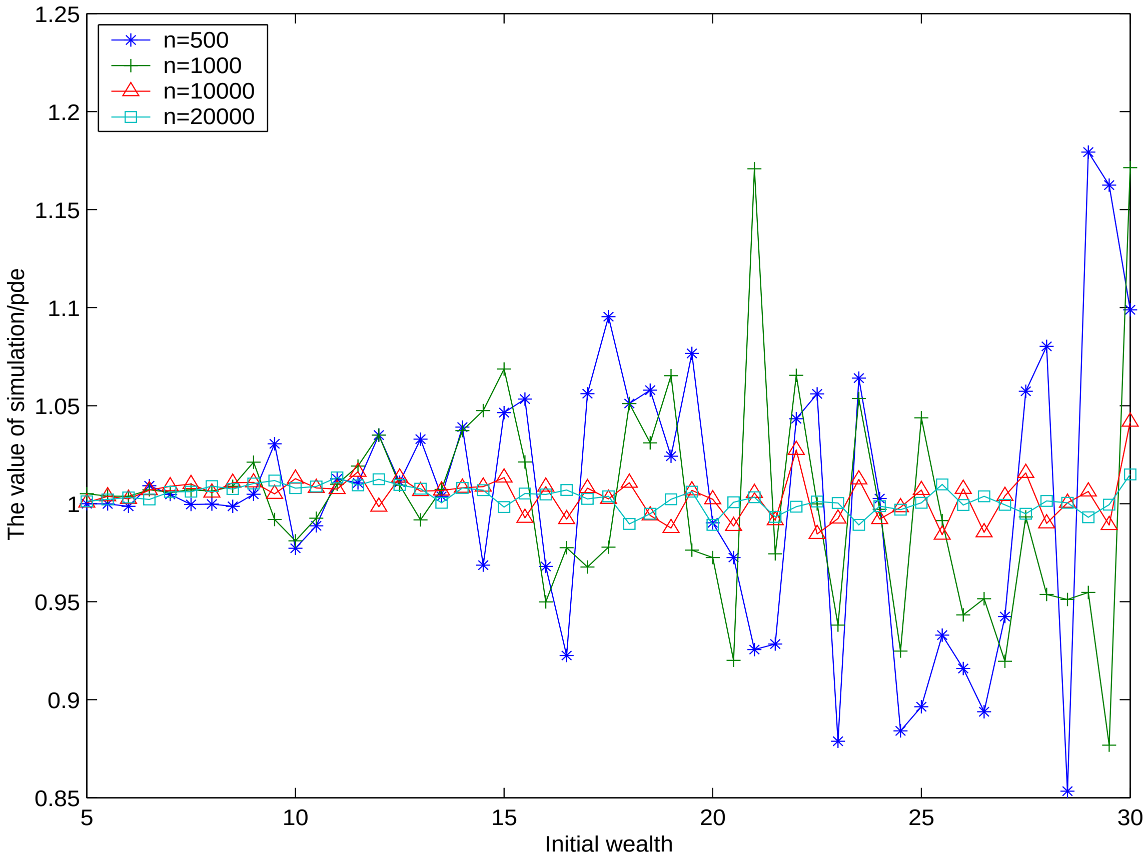

Figure 14: The figure examines the relationship between the number of simulations n, and the accuracy of the probability results for P_2 when benchmarked against the (true) PDE results. The purpose of Figure 13 is to illustrate the large number of simulations that is needed – and the implicit cost of this time – to obtain results that are relatively close to the PDE values.

Table 1: The table displays the probability that an individual with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined within 30 years (P_2) or at the end of 30 years (P_1), where ruin is defined as wealth hitting a level of y. The market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$.

y	Probability P_1	Probability P_2
0	.3517019	.3517019
1	.3639941	.3642634
2	.3749395	.3759978
3	.3875812	.390536
4	.3972156	.4028596
5	.4092463	.4207916
6	.417814	.4358777
7	.4310415	.4637261
8	.4385038	.4821129
9	.4508516	.5170691
10	.4598995	.5464339
11	.469635	.5817383
12	.474769	.6019616
13	.485585	.6483504
14	.4912722	.6748824
15	.5032159	.7356768
16	.5094758	.770427
17	.5159293	.8084472
18	.5225771	.8500499
19	.5294195	.8955809
20	.53645613	1

Table 2: The table compares the probability than an individual with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined; where ruin is defined as wealth hitting zero – within 25 years (P_2) using the exact PDE method, the Reciprocal Gamma (RG) approximation, the LogNormal (LN) approximation and the Lower Bound (CLB) estimate. The deviation of the three approximation methods from the PDE value is listed in brackets. Note that the market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$, which correspond to long-run historical values for these parameters in real (after-inflation) terms.

μ	Methods	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.5$
0.04	PDE	0.259859	0.519114	0.675381	0.861087
	RG	0.254278 (-2.15%)	0.530307 (+2.16%)	0.805383 (+19.25%)	0.999999 (+16.13%)
	LN	0.268334 (+3.26%)	0.527981 (+1.71%)	0.621477 (-7.98%)	0.720074 (-16.38%)
	CLB	0.247995 (-4.57%)	0.527702 (+1.65%)	0.665752 (+1.54%)	0.854744 (-0.74%)
0.07	PDE	0.030627	0.282757	0.514435	0.796587
	RG	0.029375 (-4.09%)	0.277442 (-1.88%)	0.578085 (+12.37%)	0.999999 (+25.54%)
	LN	0.026378 (-13.87%)	0.305567 (+8.07%)	0.492238 (-4.31%)	0.636515 (-20.09%)
	CLB	0.022586 (-44.50%)	0.285245 (+0.88%)	0.530283 (+3.08%)	0.797193 (+0.08%)
0.09	PDE	0.004070	0.163098	0.405108	0.745403
	RG	0.003537 (-13.10%)	0.157593 (-3.38%)	0.430867 (+6.36%)	0.999999 (+34.16%)
	LN	0.002096 (-48.50%)	0.179819 (+10.25%)	0.405212 (0.03%)	0.578785 (-22.35%)
	CLB	0.002087 (+51.28%)	0.159188 (-2.40%)	0.421040 (-3.93%)	0.752719 (+0.98%)
0.11	PDE	0.000360	0.083794	0.304100	0.688520
	RG	0.000241 (-33.06%)	0.080049 (-4.47%)	0.307361 (+1.07%)	0.99988 (+45.22%)
	LN	0.000066 (-81.76%)	0.089511 (+6.82%)	0.320792 (+5.49%)	0.521354 (-24.28%)
	CLB	0.000088 (-75.56%)	0.075945 (-9.37%)	0.316402 (+4.05%)	0.703814 (+2.22%)

Table 3: The table compares the probability than an individual with an initial wealth of $w = 15$ dollars (in contrast to Table 2 that examines the case of $w = 20$) who withdraws 1 dollar per annum, will get ruined; where ruin is defined as wealth hitting zero – within 25 years (P_2) using the exact PDE method, the Reciprocal Gamma (RG) approximation, the LogNormal (LN) approximation and the Lower Bound (CLB) estimate. Notice that the ruin probabilities are uniformly higher the lower the level of initial wealth. The capital market parameters are the same as Table 2.

μ	Methods	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.5$
0.04	PDE	0.659908	0.721906	0.792213	0.902623
	RG	0.677698 (+2.69%)	0.754434 (+4.50%)	0.919014 (+16.00%)	0.999999 (+10.77%)
	LN	0.680396 (+3.10%)	0.708080 (-1.92%)	0.717898 (-9.38%)	0.761642 (-15.62%)
	CLB	0.651742 (-1.24%)	0.722116 (+0.03%)	0.792457 (+0.03%)	0.891532 (-1.23%)
0.07	PDE	0.220666	0.493114	0.657118	0.851940
	RG	0.220533 (-0.06%)	0.499339 (+1.26%)	0.750392 (+14.19%)	0.999999 (+17.38%)
	LN	0.231312 (+4.82%)	0.507041 (+2.82%)	0.606400 (-7.72%)	0.684196 (-19.67%)
	CLB	0.196147 (-11.11%)	0.491684 (-0.29%)	0.663668 (+1.00%)	0.845027 (-0.81%)
0.09	PDE	0.060876	0.338484	0.553410	0.810061
	RG	0.058926 (-3.20%)	0.336481 (-0.59%)	0.609348 (+10.11%)	0.999999 (+23.45%)
	LN	0.057383 (-5.74%)	0.361978 (+6.94%)	0.525135 (-5.11%)	0.629499 (-22.29%)
	CLB	0.045223 (-25.71%)	0.332989 (-1.62%)	0.563020 (+1.74%)	0.808140 (-0.24%)
0.11	PDE	0.011049	0.209488	0.447205	0.76193
	RG	0.009542 (-13.64%)	0.205239 (-2.03%)	0.471844 (+5.51%)	0.999967 (+31.24%)
	LN	0.007144 (-35.34%)	0.229385 (+9.50%)	0.442851 (-0.97%)	0.574163 (-24.64%)
	CLB	0.005623 (-49.11%)	0.198831 (-5.09%)	0.457135 (+2.22%)	0.766686 (+0.62%)

Table 4: The table compares the probability than an individual with an initial wealth of $w = 10$ dollars (in contrast to Table 2 that examines the case of $w = 20$) who withdraws 1 dollar per annum, will get ruined; where ruin is defined as wealth hitting zero – within 25 years (P_2) using the exact PDE method, the Reciprocal Gamma (RG) approximation, the LogNormal (LN) approximation and the Lower Bound (CLB) estimate. The capital market parameters are the same as Table 2.

μ	Methods	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.5$
0.04	PDE	0.976671	0.921083	0.913555	0.947221
	RG	0.987525 (+1.11%)	0.951954 (+3.35%)	0.988115 (+8.16%)	0.887330 (-6.32%)
	LN	0.977200 (+0.05%)	0.889043 (-3.49%)	0.829773 (-9.17%)	0.814006 (-14.06%)
	CLB	0.975574 (-0.11%)	0.912405 (-0.94%)	0.903742 (-1.07%)	0.932636 (-1.54%)
0.07	PDE	0.819309	0.798142	0.833526	0.915111
	RG	0.840389 (+2.57%)	0.823716 (+3.20%)	0.924784 (+10.95%)	0.999999 (+9.28%)
	LN	0.831504 (+1.49%)	0.776120 (-2.76%)	0.749075 (-10.13%)	0.746374 (-18.44%)
	CLB	0.790777 (-3.48%)	0.783745 (-1.80%)	0.824694 (-1.06%)	0.900683 (-1.58%)
0.09	PDE	0.570904	0.674834	0.760512	0.886989
	RG	0.583943 (+2.28%)	0.689864 (+2.23%)	0.839804 (+10.43%)	0.999999 (+12.74%)
	LN	0.593166 (+3.90%)	0.670089 (-0.70%)	0.687930 (-9.54%)	0.697144 (-21.40%)
	CLB	0.512874 (-10.16%)	0.656320 (-2.74%)	0.753676 (-0.90%)	0.874459 (-1.41%)
0.11	PDE	0.288923	0.530327	0.673947	0.853074
	RG	0.291877 (+1.02%)	0.536036 (+1.08%)	0.729814 (+8.29%)	0.999999 (+17.22%)
	LN	0.303727 (+5.12%)	0.543343 (+2.45%)	0.621613 (-7.77%)	0.646185 (-24.25%)
	CLB	0.227815 (-21.15%)	0.507316 (-4.34%)	0.669148 (-0.71%)	0.844093 (-1.05%)

Table 5: Results from the CLB approximation assuming different discretization schemes. We start with the case where each period is exactly one year – which is the situation reported by Dhaene *et. al.* (2002a) – and then show the results for more quarterly, monthly and weekly compounding. Note that as one would expect, as n gets large the probabilities converge. We assume a $T = 25$ year time horizon and an initial wealth of $w = 15$. The capital market assumptions are based on historical estimates of real (after-inflation) returns, which are $\mu = 7\%$ and $\sigma = 20\%$.

n	Probability P_2
25×1	0.491684
25×4	0.504459
25×12	0.507385
25×52	0.508520
25×365	0.508817

Table 6: Limiting results from the CLB and RG approximation as T gets large. We assume an initial wealth of $w = 15$, with capital market assumptions based on historical estimates of real (after-inflation) returns, which are $\mu = 7\%$ and $\sigma = 20\%$. Thus, for example, an initial provision of 15 units would not be enough to cover 70 years of 1 unit of consumption per year, 75 percent of the time.

T	RG	CLB
25	0.499339	0.491684
30	0.577922	0.562584
35	0.631035	0.610019
40	0.667837	0.642887
50	0.712887	0.683759
60	0.737154	0.706912
70	0.750698	0.721061
∞	0.753366	